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HEAT DELIVERY IN A COMPRESSIBLE FLOW AND APPLICATIONS TO HOT-WIRE ANEMOMETRY

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HEAT DELIVERY IN A COMPRESSIBLE FLOW AND APPLICATIONS TO HOT-WIRE ANEMOMETRY

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SUMMARY

In a two-dimensional field a generalized potential theory applicable to nonadiabatic and rotational flow is developed. Three partial differential equations are first obtained determining the three variables which are: Distribution of additional temperature ϑ, velocity perturbation Φ, and an auxiliary function κ characterizing the rotationality of the flow. With the use of this theory the action of heat sources on the flow is studied, and the heat delivery in a compressible flow at subsonic and supersonic speeds is calculated. The results show the effect of compressibility and the nonlinear cooling. Applications of the results to hot-wire anemometry are discussed.

INTRODUCTION

In a two-dimensional stationary field, consider a certain distribution of heat sources placed in a horizontal stream of main velocity U, which can be subsonic or supersonic. The following points will be studied theoretically:

(1) Action of heat addition on the motion of a compressible fluid

(2) Effect of compressibility on the heat delivery

(3) Nonlinearity in the relation between heat delivery and temperature difference

For the investigations of these properties, the following problems will enter, and their analysis will form the main part of the present paper:

(1) Development of a generalized potential theory for the study of a compressible flow which is rotational and nonadiabatic (i.e., with heat addition)

(2) Temperature distribution about a heat source in a compressible flow

(3) Distribution of the flux of heat delivered by the heat sources

(4) Perturbation of the flow by the heat sources

This theoretical study will, on the one hand, illustrate the characteristic phenomena in a compressible flow with heat addition, and, on the other hand, it will find applications to hot-wire anemometry for the measurement of velocity in a compressible flow at subsonic and supersonic speeds. In order to simplify the calculations, without losing the essentiality of the problems, it will be assumed that the heat sources are distributed along a flat plate, which corresponds to an infinitely thin metallic ribbon, edgewise to the flow, so that no appreciable perturbations due to the shape of the body will be present, and that perturbations are due only to the heat sources.[1] Actually a cylindrical wire customarily is used in hot-wire anemometry for practical reasons, but from the present calculations it will be seen that the form of a flat plate is preferable to the form of a blunt body like a cylinder in the hot-wire anemometry of a compressible flow, because the variations of the Mach number with the cooling are single-valued for the former body and double-valued for the latter body. After the calculations of the heat delivery by heat sources distributed along a flat plate, the heat delivery by heat sources distributed along a blunt body will be discussed also. In the present calculations the variations of the physical constants, such as the coefficients of specific heat and the coefficient of heat conduction, as well as the effect of viscosity and radiation, will be left out of consideration.

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[1]Since the viscosity is neglected, the terminology "plate," which occurs frequently in this paper, has the only meaning of indicating the location of the heat sources, without offering any perturbation to the flow, for example, in the form of a boundary layer.

GENERALIZED POTENTIAL THEORY FOR ROTATIONAL MOTION OF A COMPRESSIBLE FLOW WITH VARIABLE TOTAL ENERGY

Fundamental Equations

By writing the equations for the motion of the fluid, the following usual symbols will be introduced:

x, y	coordinates of a variable point in two-dimensional field
$ds = \sqrt{dx^2 + dy^2}$	
w	vector velocity
u, v	components of vector velocity
p	pressure
ρ	density
T	absolute temperature
k	ratio of specific heat for constant pressure c_p and for constant volume c_v ($k = 1.4$ is taken for air in numerical calculations)
R	gas constant $\left(c_p - c_v = c_p \frac{k - 1}{k}\right)$
c	velocity of sound at temperature T $\left(\sqrt{kRT}\right)$
λ	coefficient of heat conduction
M	local Mach number (w/c)

The following functions will also be introduced:

$$\mu = 1 + \frac{k - 1}{2} M^2$$

$$\alpha = \sqrt{1 - M_\infty^2}$$

$$\beta = \mu_\infty^{-\frac{1}{k-1}}$$

$$2h = \rho_o c_p/\lambda$$

$$h' = \mu_\infty h$$

$$g = \beta h'U = \mu_\infty^{-\frac{2-k}{k-1}} hU$$

The above symbols are frequently used in the present calculations. Other ones will be introduced wherever needed. The subscript o indicates the state reduced adiabatically to stagnation, and the subscript ∞ indicates the state in the free stream.

The total temperature at a variable point is given by the definition

$$T_t = T + \frac{w^2}{2c_p} = T_o + \vartheta(x,y) \tag{1}$$

where T_o is a constant called isentropic stagnation temperature and $\vartheta(x,y)$ is the additional temperature due to an introduction of heat. For the sake of convenience the relations between the different temperature symbols are shown as follows:

$$\left.\begin{aligned} T_t/T_o &= 1 + \vartheta/T_o = \tau \\ T_t/T &= \mu \end{aligned}\right\} \tag{2}$$

Also,

$$\left.\begin{aligned} T/T_o &= \tau\mu^{-1} \\ \frac{w^2}{2c_pT_o} &= \tau(1 - \mu^{-1}) \end{aligned}\right\} \tag{3}$$

In the present problem there are four dependent variables w, p, ρ, and ϑ. For these four variables four equations will be written by expressing the following four laws for the motion of the fluid:

(1) The law of conservation of momentum

(2) The law of conservation of mass

(3) The law of conservation of energy

(4) The law of state of a perfect gas

The conservation of momentum is expressed by the equation of momentum. In vectorial form it can be written as follows:

$$1/2 \text{ grad } w^2 + \text{rot } w \times w = -\frac{1}{\rho} \text{ grad } p \tag{4}$$

In a two-dimensional problem this equation for w can be separated into two equations for u and v as follows:

$$u \frac{\partial u}{\partial x} + v \frac{\partial u}{\partial y} = -\frac{1}{\rho} \frac{\partial p}{\partial x} \tag{4a}$$

$$u \frac{\partial v}{\partial x} + v \frac{\partial v}{\partial y} = -\frac{1}{\rho} \frac{\partial p}{\partial y} \tag{4b}$$

The conservation of mass of the moving fluid is expressed by the equation of continuity

$$\text{div } w = -\frac{w}{\rho} \frac{\partial \rho}{\partial s} \tag{5}$$

where $ds = \sqrt{dx^2 + dy^2}$ is the length of a segment of the streamline.

The conservation of energy is expressed by the equation of energy, sometimes called the equation of heat conduction

$$\rho w \cdot \text{grad} \left(c_v T + \frac{1}{2} w^2 \right) = - \text{ div } (\rho w) + \lambda \text{ div grad } T$$

This equation can easily be transformed into the following one:

$$\rho w \cdot \text{grad} \left(c_p T + \frac{1}{2} w^2 \right) = \lambda \text{ div grad } T \tag{6}$$

In equation (6) T can be replaced by τ from definition (1), and the following equation can be expressed:

$$\rho = \rho_o(T/T_o)^{\frac{1}{k-1}} S_1$$

with

$$S_1 = e^{(S_o-S)/R}$$

where S is given by the definition of entropy

$$T\,dS = c_p\,dT - \frac{1}{\rho}\,dp$$

Then equation (6) becomes

$$\text{div grad } (\tau/\mu) = 2hS_1(\tau/\mu)^{\frac{1}{k-1}}\, w \cdot \text{grad } \tau \qquad (6a)$$

where $2h = \rho_o c_p/\lambda$.

Finally the equation of state of a perfect gas is

$$p = R\rho T \qquad (7)$$

These five scalar equations (equations (4a), (4b), (5), (6a), and (7)) determine the five variables in the problem u, v, p, ρ, and ϑ.

The problem is now to reduce the number of equations by eliminating certain variables. Therefore a system of three equations will be discussed which will determine the three principal variables u, v, and ϑ.

The first equation can be obtained from the equation of continuity (5), by eliminating p and ρ with the use of equations (4a), (4b), and (7) as follows:

$$\text{div } w = k\,\frac{w^2}{c^2}\,\frac{\partial w}{\partial s} + \frac{w}{T}\,\frac{\partial T}{\partial s}$$

or again, by replacing T by ϑ from equation (1),

$$\operatorname{div} w = \frac{w^2}{c^2}\frac{\partial w}{\partial s} + \frac{w}{T}\frac{\partial \vartheta}{\partial s}$$

In the x, y coordinates this equation can be written in its expanded form as follows:

$$\left(1 - \frac{u^2}{c^2}\right)\frac{\partial u}{\partial x} - \frac{uv}{c^2}\left(\frac{\partial v}{\partial x} + \frac{\partial u}{\partial y}\right) + \left(1 - \frac{v^2}{c^2}\right)\frac{\partial v}{\partial y} = \frac{1}{T}\left(u\frac{\partial \vartheta}{\partial x} + v\frac{\partial \vartheta}{\partial y}\right) \tag{8}$$

Herewith the first equation of a system of three equations is obtained which will be discussed in the following paragraphs. The first equation will be called the equation of motion of the fluid. The second equation is the equation of energy already given by equation (6), and the third equation can be derived as follows from the condition of the rotationality of the flow which is given by the equation of momentum (4). Simple calculations of the rotational term in equation (4) as a function of ϑ and T_t from equation (1) lead respectively to

$$\frac{\operatorname{rot} w \times w}{c^2} = \frac{1}{k-1}\operatorname{grad}\log_e\left(p^{\frac{1}{k}}/\rho\right) - \frac{c_p}{c^2}\operatorname{grad}\vartheta \tag{9}$$

$$\frac{\operatorname{rot} w \times w}{c^2} = -\frac{1}{k-1}\operatorname{grad}\log_e\left(p^{\frac{1}{k}}/\rho\right) - \frac{c_p}{2kR}\frac{T_t}{T}\operatorname{grad}\log_e T_t^{1/2} \tag{9a}$$

where $\operatorname{grad}\log_e\left(p^{\frac{1}{k}}/\rho\right)$ is nothing else than the increase of entropy S, expressed by

$$\operatorname{grad}\log_e\left(\frac{p^{1/k}}{\rho}\right) = \operatorname{grad}(S/c_p)$$

as can easily be proved from the definition of the entropy.

From equation (9a) it is seen that an isentropic (S = Constant) and adiabatic flow (T_t = Constant) has

$$\frac{\text{rot } w \times w}{c^2} = 0$$

Furthermore, by taking the rot of equation (9a), making use of equations (2), and observing that rot grad A = 0, there is obtained finally

$$\text{rot } \frac{\text{rot } w \times w}{c^2} = - \text{ grad } \frac{w^2}{c^2} \times \text{grad } \log_e \tau^{1/2} \tag{10}$$

This is the equation of rotationality of the flow.

The two equations (8) and (10) determine the u, v field as a function of the temperature which, in its turn, is determined by the equation of energy (6).

Generalized Potential Theory for a Rotational Flow

A system of coordinates formed by the streamlines and their orthogonal trajectories is taken. The stream function ψ is used for the streamlines, giving

$$\left.\begin{aligned} u &= \frac{\rho_o}{\rho} \frac{\partial \psi}{\partial y} \\ v &= -\frac{\rho_o}{\rho} \frac{\partial \psi}{\partial x} \end{aligned}\right\} \tag{11}$$

A <u>generalized</u> potential function φ and an auxiliary function κ can be chosen such that

$$\left.\begin{aligned} u &= \kappa \frac{\partial \varphi}{\partial y} \\ v &= \kappa \frac{\partial \varphi}{\partial y} \end{aligned}\right\} \tag{12}$$

or in vectorial form $w = \kappa \operatorname{grad} \varphi$. The function φ is called the "generalized" potential function, because it comprises the particular function for an "irrotational" potential flow by putting $\kappa = 1$. The generalized potential motion can now be determined by three equations: The equation of motion in the form of a differential equation for φ, the equation of energy which gives the temperature distribution, and a third equation which determines the auxiliary function κ.

It is worth while to remark that the study of the distribution of a certain physical property (e.g., temperature distribution) about a body of more complicated form can be made possible by the introduction of such a set of functions φ and ψ in the generalized potential theory, because the use of such variables φ and ψ as new coordinates will transform the body into a flat plate.

The third equation which determines the auxiliary function κ can be constructed as follows. If the u, v field is considered momentarily as given, the function κ will be determined by the following relations derived by simple vectorial operations:

$$\operatorname{rot} w = -w \times \operatorname{grad} \log_e \kappa \tag{13}$$

From equation (13) the rotational functions $\frac{\operatorname{rot} w \times w}{c^2}$ and $\operatorname{rot} \frac{\operatorname{rot} w \times w}{c^2}$ can be calculated as follows:

$$\frac{\operatorname{rot} w \times w}{c^2} = -\frac{w^2}{c^2} \operatorname{grad} \log_e \kappa + \frac{w \cdot \operatorname{grad} \log_e \kappa}{c^2} w \tag{14}$$

$$\operatorname{rot} \frac{\operatorname{rot} w \times w}{c^2} = -\operatorname{grad} \frac{w^2}{c^2} \times \operatorname{grad} \log_e \kappa - \frac{w \cdot \operatorname{grad} \log_e \kappa}{c^2} w \times \operatorname{grad} \log_e \kappa$$

$$w \times \operatorname{grad} \left(\frac{w \cdot \operatorname{grad} \log_e \kappa}{c^2} \right) \tag{15}$$

By comparing equation (15) with equation (10), the equation for κ is obtained in the vectorial form as follows:

$$\operatorname{grad}\frac{w^2}{c^2}\times\operatorname{grad}\log_e\kappa+\frac{w\cdot\operatorname{grad}\log_e\kappa}{c^2}\,w\times\operatorname{grad}\log_e\kappa+$$

$$w\times\operatorname{grad}\left(\frac{w\cdot\operatorname{grad}\log_e\kappa}{c^2}\right)=\operatorname{grad}\frac{w^2}{c^2}\times\operatorname{grad}\log_e\tau^{1/2} \tag{16}$$

The velocity potential φ can be written in two parts as follows:

$$\varphi=\overline{\varphi}+\Phi \tag{17}$$

where φ is the perturbed velocity potential, $\overline{\varphi}=Ux$ is the velocity potential in the free stream, and Φ is the potential perturbation. Also,

$$\left.\begin{aligned} u&=\kappa(U+\partial\Phi/\partial x)=U+u'\\ u'&=\kappa\,\partial\Phi/\partial x+(\kappa-1)U\\ v&=v'=\kappa\,\partial\Phi/\partial y \end{aligned}\right\} \tag{18}$$

where U is the velocity in the free stream.

After substitution of relations (18), equation (8) becomes

$$\left.\begin{aligned} &\left(1-\frac{u^2}{c^2}\right)\frac{\partial^2\Phi}{\partial x^2}-2\,\frac{uv}{c^2}\,\frac{\partial^2\Phi}{\partial x\partial y}+\left(1-\frac{v^2}{c^2}\right)\frac{\partial^2\Phi}{\partial y^2}=f\\ &\text{with}\\ &f=\frac{\mu}{\kappa\tau}\,w\,\frac{\partial\tau}{\partial s}-\frac{1}{\kappa^2}\left(1-M^2\right)w\,\frac{\partial\kappa}{\partial s}\\ &\text{where}\\ &ds=\sqrt{dx^2+dy^2}\\ &w=\sqrt{u^2+v^2} \end{aligned}\right\} \tag{19}$$

It may be useful to group the main equations as follows:

(a) Equation of motion (see equation (19)):

$$\left(1 - \frac{u^2}{c^2}\right)\frac{\partial^2\Phi}{\partial x^2} - 2\,\frac{uv}{c^2}\,\frac{\partial^2\Phi}{\partial x\partial y} + \left(1 - \frac{v^2}{c^2}\right)\frac{\partial^2\Phi}{\partial y^2} = f$$

where f is given in equation (19)

(b) Equation of energy:

$$\Delta\left(\mu^{-1}\tau\right) = 2h(\tau/\mu)^{\frac{1}{k-1}}\, w\,\frac{\partial\tau}{\partial s}$$

where $\Delta = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}$. This equation is the analytical form of equation (6a).

(c) Equation of rotationality (see equation (16)):

$$\operatorname{grad}\frac{w^2}{c^2} \times \operatorname{grad}\log_e \kappa + \frac{w \cdot \operatorname{grad}\log_e \kappa}{c^2}\, w \times \operatorname{grad}\log_e \kappa +$$

$$w \times \operatorname{grad}\frac{w \cdot \operatorname{grad}\log_e \kappa}{c^2} = \operatorname{grad}\frac{w^2}{c^2} \times \operatorname{grad}\log_e \tau^{1/2}$$

If s and n represent the unit vectors along a streamline and along a normal, respectively, the following equations can be written:

$$w \cdot \operatorname{grad}\log_e \kappa = w\,\frac{\partial \log_e \kappa}{\partial s}$$

$$w \times \operatorname{grad}\log_e \kappa = w\,\frac{\partial \log_e \kappa}{\partial n}$$

$$\operatorname{grad} M^2 \times \operatorname{grad}\log_e \kappa = \frac{\partial M^2}{\partial s}\,\frac{\partial \log_e \kappa}{\partial n} - \frac{\partial M^2}{\partial n}\,\frac{\partial \log_e \kappa}{\partial s}$$

Using these formulas and assuming

$$\frac{\partial \log_e \kappa}{\partial s} \ll \frac{\partial \log_e \kappa}{\partial n} \tag{20}$$

there is obtained from equation (16)

$$\kappa \approx \sqrt{\tau} \tag{21}$$

In an adiabatic flow ($\vartheta = 0$), the expression (21) reduces to the well-known value $\kappa = 1$. Assumption (20) expresses the condition that the gradient of the additional temperature (or additional energy) must be much smaller along the streamlines than along their orthogonal trajectories. This condition exists in problems of the introduction of a heat boundary (problems of combustion), of heat sources, or of a hot body in a stream moving at a high speed. In all those problems the motion of the fluid is extraneous to the heat introduction. Otherwise condition (20) will not be valid, for example, in the case of an explosion, where the predominant motion of the fluid is created by the heat emission itself.

In the present paper the problem of the rotational and nonadiabatic flow will be treated under the assumption (20) and hence with the value of κ given by expression (21). Substituting expression (21) into equation (19), the equation of motion becomes

$$\left.\begin{aligned} \left(1 - \frac{u^2}{c^2}\right)\frac{\partial^2\Phi}{\partial x^2} - 2\,\frac{uv}{c^2}\,\frac{\partial^2\Phi}{\partial x \partial y} + \left(1 - \frac{v^2}{c^2}\right)\frac{\partial^2\Phi}{\partial y^2} = f \\ \text{with} \qquad f = \frac{1}{2}\left(1 + kM^2\right)\tau^{-3/2} w \frac{\partial \tau}{\partial s} \end{aligned}\right\} \tag{22}$$

Small Perturbations

As a first approximation, equations (6a) and (22) can be transformed into linear ones by introducing the assumption of small perturbations; that is, it is assumed that the perturbations u'/U, v'/U, $\Phi/\bar{\varphi}$, and ϑ/T_o and their differentials are small compared with unity, such that terms of second order are negligible.

Under this assumption, the equation of energy (6a) and the equation of motion (22) become, respectively,

(a) Equation of energy:

$$\left.\begin{aligned} &\frac{\partial^2\vartheta}{\partial x^2} + \frac{\partial^2\vartheta}{\partial y^2} = 2g\,\frac{\partial\vartheta}{\partial x} \\ &\text{where} \\ &g = \mu_\infty^{-\frac{2-k}{k-1}} hU \end{aligned}\right\} \tag{23}$$

(b) Equation of motion:

$$\left.\begin{aligned} &\alpha^2\,\frac{\partial^2\Phi}{\partial x^2} + \frac{\partial^2\Phi}{\partial y^2} = f \\ &\text{where} \\ &\alpha = \left(1 - M_\infty^2\right)^{1/2} \\ &f = \frac{1}{2}\left(1 + kM_\infty^2\right)\frac{U}{T_o}\,\frac{\partial\vartheta}{\partial x} \end{aligned}\right\} \tag{24}$$

DISTRIBUTION OF TEMPERATURE AND HEAT FLUX FOR A HEATED BODY IN A COMPRESSIBLE FLOW

Temperature Distribution

The convection of heat is governed by equation (23). It has the following solution:

$$\vartheta = Ae^{gx}K_o(gr) \tag{25}$$

where $r = \sqrt{x^2 + y^2}$ and K_0 is a Bessel function. For larger values of gr, K_0 can be expanded into a series which will be given later in this section. The constant A can be determined if the heat delivery Q is known. Consider first the case of a line source. The formula of temperature distribution about a finite body can be constructed by considering a continuous distribution of heat sources on the body, and will be studied in the following section.

The heat delivery Q should be equal to the total heat flux around a closed circuit formed, for example, by a circle of radius r enclosing the line source. The following formula can be written:

$$Q = \int_C ds \left(-\lambda \frac{\partial \vartheta}{\partial r} + \rho c_p U \vartheta \cos \eta \right) \tag{26}$$

where ds is a segment of arc on the circle and r and η are respectively the radius and the angle in the polar coordinates.

The question is in what sector of the circuit the heat flow will be effective. At large velocities, the major part of the heat flow occurs through a very narrow sector so that the magnitude Q will be at first approximation independent of the limits of integration, provided these are taken of sufficiently large range. Therefore as a first approximation there can be written in place of formula (26), for all kinds of velocities (subsonic and supersonic),

$$Q = 2r\lambda \int_0^{\pi} d\eta \left(-\frac{\partial \vartheta}{\partial r} + 2g\mu_\infty^{-1} \vartheta \cos \eta \right) \tag{26a}$$

Putting

$$\left.\begin{aligned} I_0(z) &= \frac{1}{\pi} \int_0^{\pi} d\eta e^{z \cos \eta} \\ I_1(z) &= \frac{1}{\pi} \int_0^{\pi} d\eta e^{z \cos \eta} \cos \eta \end{aligned}\right\} \tag{27}$$

formula (26a) can be written as follows:

$$Q = 2\lambda A g r \left[-K_1(gr) \int_0^{\pi} d\eta e^{gr \cos \eta} + \left(2\mu_\infty^{-1} - 1\right) K_0(gr) \int_0^{\pi} d\eta e^{gr \cos \eta} \cos \eta \right]$$

$$= 2\pi\lambda A g r \left\{ \left[-K_1(gr) I_0(gr) + K_0 I_1 \right] - 2\left(1 - \mu_\infty^{-1}\right) K_0 I_1 \right\} \qquad (28)$$

By the use of the following expansions,

$$K_0(z) \approx e^{-z} \sqrt{\frac{\pi}{2z}} \left(1 - \frac{1}{8z}\right)$$

$$K_1(z) = \frac{dK_0}{dz} \approx -e^{-z} \sqrt{\frac{\pi}{2z}} \left(1 + \frac{3}{8z}\right)$$

$$I_0(z) \approx \frac{e^z}{\sqrt{2\pi z}} \left(1 + \frac{1}{8z}\right)$$

$$I_1(z) = \frac{dI_0}{dz} \approx \frac{e^z}{\sqrt{2\pi z}} \left(1 - \frac{3}{8z}\right)$$

formula (28) becomes

$$Q = 2\pi\lambda A \mu_\infty^{-1}$$

Herefrom A is obtained, and from formula (25) there is obtained

$$\vartheta = \frac{Q}{2\pi\lambda} e^{gx} K_0(gr) \mu_\infty \qquad (29)$$

Flux Distribution and Heat Delivery

The distribution of temperature about a line source is given by formula (29). This formula is valid for subsonic and supersonic flows. Consider now a flat plate of length $4a$. Let $q(\xi)d\xi$ represent the heat flux from an elementary segment $d\xi$ of the plate. The temperature contribution by this segment at a variable point x,y is

$$d\vartheta = \frac{\mu_\infty}{2\pi\lambda} d\xi q(\xi) e^{g(x-\xi)} K_o\left(g\sqrt{z^2 + y^2}\right)$$

where $z = x - \xi$. By integrating,

$$\vartheta(x,y) = \frac{\mu_\infty}{2\pi\lambda} \int_0^{4a} d\xi q(\xi) e^{g(x-\xi)} K_o\left(g\sqrt{z^2 + y^2}\right) \tag{30}$$

Formula (30) gives the distribution of temperature in the flow region about a plate. This formula contains a function q which will be determined, in the calculations which follow formula (30), by the integral equation (31) for the surface condition on the plate.

On the surface of the plate there can be written for $0 \leqq x \leqq 4a$

$$\vartheta_w = \frac{\mu_\infty}{2\pi\lambda} \int_0^x d\xi q(\xi) e^{g(x-\xi)} K_o\left[g(x - \xi)\right] +$$

$$\frac{\mu_\infty}{2\pi\lambda} \int_x^{4a} d\xi q(\xi) e^{-g(\xi-x)} K_o\left[g(\xi - x)\right] \tag{31}$$

or, by using the asymptotic expansion of $K_o(z)$ for large values of z,

$$\vartheta_w = \frac{\mu_\infty}{2\pi\lambda} \int_0^x d\xi q(\xi) \sqrt{\frac{\pi}{2g(x - \xi)}} + \frac{\mu_\infty}{2\pi\lambda} \int_x^{4a} d\xi q(\xi) e^{-2g(\xi-x)} \sqrt{\frac{\pi}{2g(\xi - x)}} \tag{31a}$$

The second integral on the right-hand side decreases rapidly with ξ, and therefore as a first approximation it can be written as follows:

$$\int_x^{4a} d\xi q(\xi) e^{-2g(\xi-x)} \sqrt{\frac{\pi}{2g(\xi - x)}} \approx q(x)\sqrt{\frac{\pi}{2g}} \int_0^{\infty} d\xi_1 e^{-2g\xi_1} \frac{1}{\sqrt{\xi_1}}$$

$$= q(x) \frac{\pi}{2g}$$

For the sake of simplification of writing, introduce

$$\left.\begin{aligned} q_1(x) &= \frac{1}{g_*} \frac{\mu_\infty}{2\pi\lambda} q(x) \\ \text{where} \qquad g_* &= \sqrt{2g/\pi} \end{aligned}\right\} \tag{31b}$$

The relation (31a) between the heat flux and the wall temperature then becomes

$$\vartheta_w = \int_0^x d\xi \frac{q_1(\xi)}{\sqrt{x - \xi}} + g_*^{-1} q_1(x)$$

or

$$g_* \vartheta_w = q_1(x) + g_* \int_0^x d\xi \frac{q_1(\xi)}{\sqrt{x - \xi}} \tag{32}$$

First the integral equation (32) will be solved for the general case of a variable wall temperature ϑ_w. Later the particular case will be derived of a constant ϑ_w which is true, if the plate is supposed to be a very good conductor.

The integral equation will be reduced to a linear differential equation of first order by a differentiation with respect to x. First, put $x - \xi = \zeta^2$ in order to prevent the singularity which would occur when $\xi = x$. The differential with respect to x of the integral in the right-hand member of equation (32) is

$$\frac{d}{dx}\int_0^x d\xi \frac{q_1(\xi)}{\sqrt{x - \xi}} = 2\frac{d}{dx}\int_0^{\sqrt{x}} d\zeta q_1(x - \zeta^2)$$

$$= \frac{q_1(x - \zeta^2)}{\sqrt{x}}\bigg|_{\zeta=\sqrt{x}} + 2\int_0^{\sqrt{x}} d\zeta \dot{q}_1(x - \zeta^2)$$

$$= \frac{q_1(0)}{\sqrt{x}} + \int_0^x d\xi \frac{\dot{q}_1(\xi)}{\sqrt{x - \xi}}$$

where $\dot{q}_1(\xi) = \frac{dq(\xi)}{d\xi}$.

By applying this rule to the differentiation of equation (32), there is obtained

$$\dot{q}_1(x) + g_* \frac{q_1(0)}{\sqrt{x}} + g_* \int_0^x d\xi \frac{\dot{q}_1(\xi)}{\sqrt{x - \xi}} = g_* \dot{\vartheta}_w \tag{33}$$

This formula is fulfilled by the flux at the distance x. The flux at the distance x_2 will fulfill

$$\dot{q}_1(x_2) + g_* \frac{q_1(0)}{\sqrt{x_2}} + g_* \int_0^{x_2} d\xi \frac{\dot{q}_1(\xi)}{\sqrt{x_2 - \xi}} = g_* \dot{\vartheta}_w(x_2) \tag{34}$$

Multiply every term of equation (34) by $dx_2 g_*(x - x_2)^{-1/2}$, integrate with respect to x_2 between the limits 0 and x, and subtract from equation (33); it follows:

$$\dot{q}_1(x) + g_* \frac{q_1(0)}{\sqrt{x}} - g_*^2 q_1(0) \int_0^x dx_2 \frac{1}{\sqrt{x_2(x - x_2)}} -$$

$$g_*^2 \int_0^x dx_2 \int_0^{x_2} d\xi \frac{\dot{q}_1(\xi)}{\sqrt{(x - x_2)(x_2 - \xi)}} = g_* \dot{\vartheta}_w - g_*^2 \int_0^x dx_2 \frac{\dot{\vartheta}_w(x_2)}{\sqrt{x - x_2}} \tag{35}$$

The first integral on the left-hand side of this equation has the value π. In the double integral the order of the integration can be inverted as follows:

$$\int_0^x dx_2 \int_0^{x_2} d\xi \frac{\dot{q}_1(\xi)}{\sqrt{(x - x_2)(x_2 - \xi)}} = \int_0^x d\xi \dot{q}_1(\xi) \int_\xi^x dx_2 \frac{1}{\sqrt{(x - x_2)(x_2 - \xi)}}$$

The last integral with respect to x_2 has the value π. Hence the value of the double integral becomes

$$\int_0^x dx_2 \int_0^x d\xi \frac{\dot{q}(\xi)}{\sqrt{(x - x_2)(x_2 - \xi)}} = \pi \left[q_1(x) - q_1(0) \right]$$

Substituting this into equation (35),

$$\left.\begin{aligned} &\dot{q}_1(x) - 2gq_1(x) = F(x) \\ \text{where} \quad &F(x) = -g_*q_1(0)x^{-1/2} + g_*^2\dot{\vartheta}_w - g_*^2\int_0^x dx_2 \frac{\dot{\vartheta}_w(x_2)}{\sqrt{x - x_2}} \end{aligned}\right\} \tag{36}$$

The integral equation (32) is thus transformed into the linear differential equation (36). Its solution is

$$\begin{aligned} q_1(x) &= q_1(0)e^{2gx} + \int_0^x d\xi F(\xi)e^{2g(x-\xi)} \\ &= q_1(0)e^{2gx}\left(1 - \operatorname{erf}\sqrt{2gx}\right) + q_1'(x) \end{aligned} \tag{37}$$

where

$$\operatorname{erf}\sqrt{2gx} = \frac{2}{\sqrt{\pi}}\int_0^{\sqrt{2gx}} dx e^{-x^2}$$

and $q_1'(x)$ is the flux due to the gradient of the wall temperature

$$\left.\begin{aligned} &q_1'(x) = \int_0^x d\xi F'(\xi)e^{2g(x-\xi)} \\ \text{with} \quad &F'(\xi) = g_*\dot{\vartheta}_w - g_*^2\int_0^x dx_2 \frac{\dot{\vartheta}_w(x_2)}{\sqrt{x - x_2}} \end{aligned}\right\} \tag{38}$$

In the particular case of a plate which is a good conductor, the wall temperature can be assumed constant; then formula (37) becomes simply

$$q_1(x) = q_1(0)e^{2gx}\left(1 - \operatorname{erf}\sqrt{2gx}\right) \tag{37a}$$

or, by returning to q,

$$q(x) = q(0)e^{2gx}\left(1 - \operatorname{erf}\sqrt{2gx}\right)$$

This relation is plotted in figure 1, where $\sqrt{2gx}$ is denoted by u, which can be expressed as

$$u = \sqrt{P}\mu_\infty^{-\frac{2-k}{2(k-1)}}\sqrt{\frac{x}{4a}}$$

where P is the Peclet number as will be defined later.

The constant $q_1(0)$ in formula (37a) can be determined by putting $x = 0$ in equation (32) which gives:

$$q_1(0) = \frac{\mu_\infty}{2\pi\lambda g_*}\,q(0) = g_*\vartheta_w \tag{37b}$$

Now pass over to the calculation of the heat delivery. By definition,

$$Q_1 = \int_0^{4a} dx q_1(x) = \int_0^{4a} dx q_1(0)e^{2gx}\left(1 - \operatorname{erf}\sqrt{2gx}\right)$$

or

$$Q_1/q_1(0) = \int_0^{4a} dx e^{2gx} - \int_0^{4a} dx e^{2gx} \operatorname{erf}\sqrt{2gx}$$

The first integral of the right-hand member is evident. By changing the variables $\eta^2 = 2gx$, the second integral of the right-hand member can be written as follows:

$$\int_0^{4a} dx e^{2gx} \operatorname{erf} \sqrt{2gx} = \frac{2}{\sqrt{\pi}g} \int_0^{\sqrt{8ga}} d\eta \eta e^{\eta^2} \int_0^{\eta} d\xi e^{-\xi^2}$$

$$= \frac{2}{\sqrt{\pi}g} \int_0^{\sqrt{8ga}} d\xi e^{-\xi^2} \int_{\xi}^{8ga} d\eta \eta e^{\eta^2}$$

$$= \frac{1}{2g} e^{8ga} \operatorname{erf} \sqrt{8ga} - \frac{1}{g\sqrt{\pi}} \sqrt{8ga}$$

Hence,

$$\frac{Q}{q(0)} = \frac{Q_1}{q_1(0)} = \frac{1}{2g}\left[\left(\frac{2}{\sqrt{\pi}} \sqrt{8ga} - 1\right) + e^{8ga}\left(1 - \operatorname{erf} \sqrt{8ga}\right)\right] = \frac{\sigma_1}{2g} \quad (39)$$

where, for $ga \gg 1$, σ_1 is

$$\sigma_1 = \frac{2}{\sqrt{\pi}} \sqrt{8ga} \quad (39a)$$

Define a Nusselt number N such that

$$Q = N\lambda A \vartheta_w / 4a$$

where A is the area per unit length $8a$. Hence,

$$N = \frac{Q}{2\lambda \vartheta_w} \quad (40)$$

From expression (37b),

$$q(0) = 2\lambda\vartheta_w \frac{2g}{\mu_\infty} \tag{41}$$

Hence,

$$N = \frac{Q}{2\lambda\vartheta_w} = \sigma_1/\mu_\infty = \frac{2}{\pi}\sqrt{8\pi ga}\Big/\mu_\infty \tag{42}$$

The above calculations give a picture of the distribution of the heat flux. However, in the order of approximation adopted, it seems that the pattern of distribution of the heat flux does not influence very much the heat delivery. This is readily seen, since the expression for the heat delivery (formula (42)) can also be obtained by the following more approximate method which does not take into account the details of the flux distribution. To this end, if in equation (31) a mean heat flux $\bar{q}$ is introduced and x is put equal to $4a$, the following equation can be written:

$$\vartheta_w \approx \frac{\mu_\infty}{2\pi\lambda}\bar{q}\int_0^{4a} d\xi e^{g(4a-\xi)}K_o\left[g(4a-\xi)\right]$$

$$= \frac{\mu_\infty}{2\pi\lambda}\bar{q}g^{-1}\int_0^{4ga} du e^u K_o(u) \tag{42a}$$

This integral is known and has the value $\sqrt{8\pi ga} - 1$. Then,

$$\bar{q} = 2\lambda\vartheta_w \pi g \mu_\infty^{-1}\left(\sqrt{8\pi ga} - 1\right)^{-1}$$

and N which is defined by

$$\frac{4a\bar{q}}{2\lambda\vartheta_w}$$

becomes

$$N = \frac{1}{2\mu_\infty}\left(\sqrt{8\pi ga} + 1\right) \tag{43}$$

By comparing formulas (42) and (43), it is seen that formula (43) gives a very good approximation to formula (42) which was obtained from the considerations of the exact distribution of heat flux.

A generalized potential theory has been developed and three differential equations have been obtained which determine the three variables: The distribution of temperature difference ϑ, the potential perturbation Φ, and the auxiliary function of rotationality κ. As a first approximation, the equation has been linearized by neglecting terms of the second order in perturbations u'/U, $\Phi/\bar{\varphi}$, and ϑ_w/T_o (see section Small Perturbations). Then the equations can be solved for ϑ (section Temperature Distribution), the heat flux can be calculated (this section), and hence the heat delivery can be obtained as in formula (43). However, the results show only a linear relation between the heat delivery and the temperature difference. In order to show the nonlinear character, a second approximation must be pursued by keeping the second-order terms of perturbations in the calculations of heat delivery. To this end, the distribution of temperature difference ϑ obtained in the first approximation will be used to calculate first the potential perturbation Φ (two following sections), and herefrom the velocity perturbation u'/U. An "effective velocity" which is the average perturbed velocity of the fluid along the plate can be introduced in the expression of the heat delivery which will then be of nonlinear nature (see first section under Nonlinear Cooling of a Heated Body at Supersonic Speed).

It is to be remarked that the process mentioned above will give a second approximation with sufficient accuracy without entering into the integration of the nonlinear partial differential equation in the form of equation (6a).

ACTION OF HEAT ADDITION ON MOTION OF A COMPRESSIBLE FLOW

Flow Perturbations by Heat Sources at Subsonic Speed

In this section the flow perturbation at a subsonic speed by heat sources distributed along a flat plate will be investigated by using equation (24). By means of the transformation

$$\left.\begin{aligned} x_i &= x \\ y_i &= \alpha y \\ \Phi_i &= \nu\Phi \end{aligned}\right\} \tag{44}$$

equation (24) can be brought to a Poisson's equation of the form

$$\frac{\partial^2\Phi_i}{\partial x_i^2} + \frac{\partial^2\Phi_i}{\partial y_i^2} = f\nu\alpha^{-2} \tag{45}$$

For the sake of mathematical convenience this equation for the motion of the fluid about a plate can again be transformed by conformal representation into an equation for the motion of the fluid about a cylinder of radius a, using the formula of conformal transformation

$$z = Z + \frac{a^2}{Z}$$

where $z = x_i + iy_i$ represents the plane of the flow about the plate, and $Z = X + iY$ the plane of the flow about the cylinder. By transforming equation (45) into an equation for a cylinder, there is obtained the following equation:

$$\frac{\partial^2\Phi_i}{\partial X^2} + \frac{\partial^2\Phi_i}{\partial Y^2} = f\nu a^{-2}\left|\frac{dz}{dZ}\right|^2 = f_i \tag{46}$$

This equation can be made dimensionless by dividing X and Y by a, and Φ_i by 2Ua. When R and η are taken as the polar coordinates of the Z-plane, and

$$\left|\frac{dz}{dZ}\right|^2 = 1 - 2R^{-2}\cos 2\eta + R^{-4}$$

the right-hand member of equation (46) becomes

$$f_i = \frac{1}{4}\nu\alpha^{-2}\left(1 + kM_\infty{}^2\right)\left(1 - 2R^{-2}\cos 2\eta + R^{-4}\right)\frac{\partial\vartheta/T_o}{\partial x} \qquad (47)$$

where $\partial\vartheta/\partial x$ should be obtained by differentiating expression (30) for the temperature distribution. However, the latter expression's being of integral form will make the Poisson's equation (46) rather cumbersome to handle. As a convenient but not as a necessary step, the perturbation function Φ_i is considered as a superposition of elementary perturbations corresponding to elementary heat sources. First calculate the elementary perturbation produced by an elementary heat source from equation (46). An integration according to the proper distribution of heat sources will give us the resultant perturbation.

Following the above procedure, consider first a heat source of intensity Q, placed at the origin of the coordinate (center of the plate); then $\partial\vartheta/\partial x$ can be calculated from formula (29), and expression (47) can be written as follows:

$$f_i \approx B_1\nu\frac{1}{\sqrt{gar_i}}e^{-gar_i(1-x_i/r_i)}(1 - x_i/r_i)\left|\frac{dz}{dZ}\right|^2 \qquad (47a)$$

where

$$B_1 = \frac{1}{4\sqrt{2\pi}}\alpha^{-2}\left(1 + kM_\infty{}^2\right)\mu_\infty\frac{\vartheta_w}{T_o}\frac{Q}{2\lambda\vartheta_w}ga$$

It is worth while to transform this formula into the polar coordinates R and η of the Z-plane. To this end, note first that the exponential term in the function f_i can be reduced to the form

$$f_i \sim e^{-\text{constant}\times ga(R-1)^2}$$

Since large values of ga must be dealt with, the function f_i keeps a significant value only for $\zeta = R - 1 \ll 1$. Therefore the approximation $\zeta \ll 1$ can be introduced in the transformation process. Under such an approximation, the variables x_i and y_i in terms of ζ and η become

$$x_i = 2 \cos \eta$$

$$y_i = 2\zeta \sin \eta$$

with

$$r_i = \sqrt{x_i^2 + y_i^2}$$

It follows that, at a variable point downstream of the source for $-\pi/2 \ll \eta \ll + \pi/2$,

$$f_i = B_1 \nu \frac{2}{\sqrt{2ga \cos \eta}} \left(\frac{\sin^2 \eta}{\cos \eta} \right)^2 \zeta^2 e^{-ga \frac{\sin^2 \eta}{\cos \eta} \zeta^2} \tag{47b}$$

Now the integration of Poisson's equation (46) for the Z-plane will be considered. It is to be remarked that the effect of the presence of a right-hand member in equation (46) is equivalent to that of a certain continuous distribution of sources throughout the region of flow, the intensity dI of the source distribution in an element of area $l \, dl \, d\chi$ being

$$dI = \frac{f_i}{2\pi} l \, dl \, d\chi$$

where l and χ are the polar coordinates. The potential function induced by this distribution of sources can be calculated by the method of image (references 1 to 3).

Consider first a unit source placed at a point $Z_Q = le^{i\chi}$ in the flow field outside the cylinder. The velocity potential induced by this source at a variable point $Z = Re^{i\eta}$ is equivalent to the velocity potential of the unit source placed at Z_Q, its unit image placed at the point $Z_r = l^{-1}e^{i\chi}$ reciprocal to Z_Q, and a unit source of intensity -1 placed at the center of the cylinder. When the potential function and

stream function of the perturbation induced by the unit source are denoted by Φ^* and Ψ^*, respectively, there can be written

$$\Phi^* + i\Psi^* = \log_e \left(Z - Z_Q\right) + \log_e \left(Z - Z_r\right) - \log_e Z$$

By taking the real part of the above complex equation, there is obtained

$$\Phi^* = \frac{1}{2} \log_e \left(l^2 - 2lR \cos (X - \eta) + R^2\right) +$$

$$\frac{1}{2}\left[1 - 2(lR)^{-1} \cos (X - \eta) + (lR)^{-2}\right]$$

Evidently this expression can be written in the two following identical forms:

$$\Phi^* = \frac{1}{2} \log_e l^2 + \frac{1}{2} \log_e \left[1 - 2\xi_1 \cos (X - \eta) + \xi_1^2\right] +$$

$$\frac{1}{2} \log_e \left[1 - 2\xi_2 \cos (X - \eta) + \xi_2^2\right]$$

$$\equiv \frac{1}{2} \log_e R^2 + \frac{1}{2} \log_e \left[1 - 2\xi_1' \cos (X - \eta) + (\xi_1')^2\right] +$$

$$\frac{1}{2} \log_e \left[1 - 2\xi_2 \cos (X - \eta) + \xi_2^2\right] \qquad (48)$$

where, for the sake of simplification,

$$\xi_1 = R/l$$

$$\xi_1' = l/R$$

$$\xi_2 = (lR)^{-1}$$

The total induced velocity potential can be obtained by integrating Φ^* for the whole region outside the cylinder as follows:

$$\Phi_i = \frac{1}{2\pi}\int_1^\infty dl \int_0^{2\pi} d\chi l f_i(l,\chi)\Phi^*(l,\chi) \tag{49}$$

Thus the solution of Poisson equation (46) is reduced to the solution of the double integral (49).

Substituting expression (48) into formula (49), the double integral can be written as follows:

$$\Phi_i = \frac{1}{2\pi}\int_1^R dl \int_0^{2\pi} d\chi l f_i \left\{\frac{1}{2}\log_e R^2 + \frac{1}{2}\log_e \left[1 - 2\xi_1' \cos(\chi - \eta) + \xi_1'^2\right] + \frac{1}{2}\log_e \left[1 - 2\xi_2 \cos(\chi - \eta) + \xi_2^2\right]\right\} +$$

$$\frac{1}{2\pi}\int_R^\infty dl \int_0^{2\pi} d\chi l f_i \left\{\frac{1}{2}\log_e l^2 + \frac{1}{2}\log_e \left[1 - 2\xi_1 \cos(\chi - \eta) + \xi_1^2\right] + \frac{1}{2}\log_e \left[1 - 2\xi_2 \cos(\chi - \eta) + \xi_2^2\right]\right\} \tag{50}$$

Only the velocity perturbation at the surface of the cylinder (R = 1) is of interest; this is

$$\Phi_{io} = \frac{1}{2\pi}\int_1^\infty dl \int_0^{2\pi} d\chi l f_i \log_e l + \frac{1}{2\pi}\int_1^\infty dl\, l \int_0^{2\pi} d\chi f_i \log_e \left[1 - 2\xi \cos(\chi - \eta) + \xi^2\right] \tag{50a}$$

with $\xi = 1/l$.

It can be proved that for a flat plate the perturbation upstream of the heat source decreases very rapidly with the distance, in the

proportion $e^{-2ga|x_i|}$, and hence it is negligible with respect to the perturbation downstream. This means that the variations of $f_i(l,\chi)$ have to be restricted to the half plane corresponding to $-\pi/2$ and $\pi/2$. Therefore the limits of integration in formula (50a) will be $-\pi/2$ and $\pi/2$. In order to carry out the integrations of the right-hand member of formula (50a), the following simplifications can be introduced:

(1) The variable l can be changed into $1 + \zeta$ with $\zeta \ll 1$

(2) Since $\log_e l \approx \zeta$, the first double integral of the right-hand member of formula (50a) can be neglected in comparison with the second one

(3) The order of integrations in the second double integral of the right-hand member can be interchanged; by first integrating f_i with respect to ζ,

$$\int_0^x d\zeta f_i(l,\chi) = B_1 \nu \frac{\sqrt{\pi}}{2\sqrt{2}} (ga)^{-2} tg\chi$$

Hence formula (50a) is reduced to

$$\Phi_{io} \approx B_1 \nu \frac{1}{4\sqrt{2\pi}} (ga)^{-2} \int_{-\pi/2}^{\pi/2} d\chi \log_e \left\{ 2\left[1 - \cos(\chi - \eta)\right] \right\} tg\chi$$

$$= B_1 \nu \frac{1}{4\sqrt{2\pi}} (ga)^{-2} \int_0^{\pi/2} d\chi \frac{\sin^2\chi}{\cos\chi} \frac{1}{\sin\chi} \log_e \frac{1 - \cos(\chi - \eta)}{1 - \cos(\chi + \eta)}$$

As the logarithmic term has predominant absolute values in the proximity of $\chi \sim \eta$, there can be written by approximation

$$\Phi_{io} \approx B_1 \nu \frac{1}{4\sqrt{2\pi}} (ga)^{-2} \frac{\sin^2\eta}{\cos\eta} \int_0^{\pi/2} d\chi \frac{1}{\sin\chi} \log_e \frac{1 - \cos(\chi - \eta)}{1 - \cos(\chi + \eta)}$$

Now the integral on the right-hand side has the following limit values:

(1) For $\eta = 0$ it is zero

(2) For $\eta = \frac{\pi}{2}$ it is

$$\int_0^{\pi/2} d\chi \, \frac{1}{\sin \chi} \log_e \frac{1 - \sin \chi}{1 + \sin \chi} = -\frac{\pi^2}{2}$$

(See reference 4.) For intermediary values of η, the approximate form $-\frac{\pi^2}{2} \sin \eta$ is suggested. Hence,

$$\Phi_{io} \approx -B_1 \nu \, \frac{\pi\sqrt{\pi}}{8\sqrt{2}} (ga)^{-2} \frac{\sin^3 \eta}{\cos \eta} \tag{51}$$

The distribution of the potential perturbation Φ_o on the compressible field along a plate of length $4a$ can be obtained by multiplying formula (51) by $2Ua$, by using relations (44), and by replacing $\cos \eta$ by $x/2a$; then,

$$\Phi_o = -B_1 \, \frac{\pi\sqrt{\pi}}{4\sqrt{2}} (ga)^{-2} Ua \, \frac{2a}{x} \left[1 - (x/2a)^2\right]^{3/2}$$

or, by substituting for B_1,

$$\Phi_o = -\frac{\pi}{8} B_2 U \mu_\infty g^{-1} \, \frac{Q}{2\lambda \vartheta_w} \, \frac{2a}{x} \left[1 - (x/2a)^2\right]^{3/2} \tag{52}$$

where B_2 represents for the sake of simplification

$$B_2 = \frac{1}{4} \alpha^{-2} (1 + kM_\infty^2) \, \frac{\vartheta_w}{T_o} \tag{53}$$

Formula (52) gives the potential perturbation by a single heat source at a point x from the source.

Now consider a continuous distribution of heat sources such that the intensity in the region between ξ and $\xi + d\xi$ from the leading edge of the plate is $q(\xi)\, d\xi$. Thus the elementary perturbations by $q(\xi)\, d\xi$ along the downstream part of the segment of plate which has a length $4a - \xi$ can be obtained from formula (52) by replacing $2a$ by $4a - \xi$ and x by $x' - \xi$, where x' is the distance from the leading edge. For $x' - \xi \geqq 0$,

$$d\Phi_o(\xi) = -\frac{\pi}{8} B_2 U \mu_\infty g \frac{q(\xi)\, d\xi}{2\lambda\vartheta_w} \frac{4a - \xi}{x' - \xi}\left[1 - \left(\frac{x' - \xi}{4a - \xi}\right)^2\right]^{3/2}$$

The total downstream perturbation corresponding to a continuous distribution of heat sources along the whole plate is obtained by an integration of the above expression with respect to ξ between the limits 0 and x' as follows:

$$\Phi_o = -\frac{\pi}{4} B_2 U \frac{\mu_\infty}{2g} \frac{1}{2\lambda\vartheta_w} \int_0^{x'} d\xi q(\xi) \frac{4a - \xi}{x' - \xi}\left[1 - \left(\frac{x' - \xi}{4a - \xi}\right)^2\right]^{3/2} \qquad (54)$$

In order to calculate the integral in the right-hand member of the above expression, the following simplifications are introduced:

(1) The asymptotic form of the formula (37a) for flux distribution will be used

$$q(\xi) \approx q(0) \frac{1}{\sqrt{2\pi g \xi}}$$

where

$$q(0) = \left(\frac{\mu_\infty}{2g} \frac{1}{2\lambda\vartheta_w}\right)^{-1}$$

(2) A change of variables $\frac{x' - \xi}{4a - \xi} = \cos u$ transforms expression (54) into the following type:

$$\Phi_o \sim \int_{\cos^{-1} v}^{\pi/2} d\eta \frac{\sin \eta}{\sqrt{v - \cos u}} F(\eta)$$

where $F(\eta)$ is a trigonometrical function which appears in the course of the transformation and v is a parameter which turns out to be $v = x'/4a$. The detailed form of $F(\eta)$ will not be written out here. As the fractional term has predominant values in the proximity of $\cos u \approx v$, there can be written by approximation

$$\int_{\cos^{-1}v}^{\pi/2} d\eta \frac{\sin}{\sqrt{v - \cos u}} F(\eta) \approx F(v) \int_{\cos^{-1}v}^{\pi/2} d\eta \frac{\sin \eta}{\sqrt{v - \cos u}}$$

The integral on the right-hand side can be immediately calculated, and there is obtained

$$\Phi_o = -\pi B_2 \frac{Ua}{\sqrt{2\pi ga}} v^{-1/2}(1 - v)(1 + v)^{3/2}$$

The gradient at the center of the plate will be taken as the effective gradient ϵ which is

$$\epsilon = \frac{1}{U}\frac{d\Phi_o}{dx} = \frac{3\sqrt{3}}{4}\sqrt{\pi}B_2 \frac{1}{\sqrt{8ga}}$$

$$= \frac{3\sqrt{3}}{16}\sqrt{\pi}\alpha^{-2}(1 + kM_\infty^2)\frac{\vartheta_w}{T_o}\frac{1}{\sqrt{8ga}} \qquad (55)$$

It is seen that in a subsonic field the fluid increases its velocity by passing over the heat sources.

Flow Perturbations by Heat Sources at Supersonic Speed

In this section the flow perturbations at a supersonic speed by heat sources distributed along a flat plate will be investigated by using again equation (24). The equation will be first solved for a single heat source placed at the origin. The perturbations of the flow about a flat plate will be derived by a superposition of heat sources as done in the preceding section. Equation (24) can be made dimensionless by dividing the coordinates by a, and the potential perturbation by $2Ua$,

where $4a$ is the length of the plate and U is the free-stream velocity. Thus,

$$\left.\begin{aligned} &\frac{1}{\sigma^2}\frac{\partial^2\Phi}{\partial x^2} - \frac{\partial^2\Phi}{\partial y^2} = -f \\ &\text{where} \\ &f = \frac{1}{4}\left(1 + kM_\infty{}^2\right)\frac{\partial \vartheta/T_o}{\partial x} \\ &\sigma = \left(M_\infty{}^2 - 1\right)^{-1/2} = \operatorname{tg}\eta_o \\ &\eta_o = \text{Mach angle} \end{aligned}\right\} \qquad (56)$$

By means of a change of variables

$$\left.\begin{aligned} x_i &= \sigma x \\ y_i &= y \end{aligned}\right\} \qquad (57)$$

equation (56) can be transformed into the following form:

$$\frac{\partial^2\Phi}{\partial y_i{}^2} - \frac{\partial^2\Phi}{\partial x_i{}^2} = f \qquad (56a)$$

Equation (56a) has its characteristic lines given by $y_i \pm x_i = C$; these are two families of straight lines making an angle of 45° with the x_i-axis.

The problem is to find a solution of equation (56a) such that it satisfies the following two boundary conditions:

(a) The term $\partial\Phi/\partial y_i$ vanishes at $y_i = 0$

(b) The perturbation Φ vanishes along the front shock waves, which are formed by the characteristic lines originating at the leading edge.

In particular, the solution at various distances x_i on the x_i-axis is of special interest.

The method of characteristics will be used for the integration of equation (56a). Therefore, take a point x_i on the x_i-axis, and draw a line x_ip parallel to $0\eta'$ (see fig. 2). In the plane x_iy_i apply the Gaussian theorem in the form

$$\iint_{0x_ip} dx_i\, dy_i\left(\frac{\partial V}{\partial x_i} - \frac{\partial U}{\partial y_i}\right) = \oint (U\, dx_i + V\, dy_i) \tag{58}$$

where the double integration is extended over the area $0x_ip$, and the line integral along the contour of this area in the positive direction. If in equation (58) there are inserted $U = \partial\Phi/\partial y_i$ and $V = \partial\Phi/\partial x_i$, it follows from equations (56a) and (58) that

$$\oint\left(\frac{\partial\Phi}{\partial y_i}\, dx_i + \frac{\partial\Phi}{\partial x_i}\, dy_i\right) = -\iint_{0x_ip} dx_i\, dy_i f \tag{59}$$

It is evident that $dx_i = dy_i$ on the line $0p$, and $dx_i = -dy_i$ on the line x_ip. Consequently,

$$\int_0^p\left(\frac{\partial\Phi}{\partial x_i}\, dy_i + \frac{\partial\Phi}{\partial y_i}\, dx_i\right) = \int_0^p\left(\frac{\partial\Phi}{\partial x_i}\, dx_i + \frac{\partial\Phi}{\partial y_i}\, dy_i\right) = \Phi_p - \Phi_o$$

$$\int_{x_i}^p\left(\frac{\partial\Phi}{\partial x_i}\, dy_i + \frac{\partial\Phi}{\partial y_i}\, dx_i\right) = -\int_{x_i}^p\left(\frac{\partial\Phi}{\partial x_i}\, dx_i + \frac{\partial\Phi}{\partial y_i}\, dy_i\right) = -\Phi_p + \Phi x_i$$

$$\int_0^{x_i}\left(\frac{\partial\Phi}{\partial x_i}\, dy_i + \frac{\partial\Phi}{\partial y_i}\, dx_i\right) = \int_0^{x_i} dx_i\, \frac{\partial\Phi}{\partial y_i}$$

and it follows from equation (59) that

$$\Phi x_i - 2\Phi_p + \Phi_o + \int_0^{x_i} dx_i \frac{\partial \Phi}{\partial y_i} = -\iint_{0x_ip} dx_i \, dy_i f$$

Now, from the boundary conditions,

$$\left(\frac{\partial \Phi}{\partial y_i}\right)_{y_i=0} = 0$$

$$\Phi_p = \Phi_o = 0$$

Hence,

$$\Phi x_i = -\iint_{0x_ip} dx_i \, dy_i f \tag{60}$$

The above formula can be transformed from the x_i,y_i-plane into the x,y-plane according to relations (57). Omitting the details, the result is

$$\Phi_x = -\frac{2}{\sigma} \int_0^{x/2} dx_1 \int_0^{x_1} dy_1 f(x_1 y_1) \tag{61}$$

where $1/\sigma$ is the ratio between the two areas of integration in the respective planes

$$\frac{1}{\sigma} = \frac{\partial(x,y)}{\partial(x_i,y_i)}$$

For a heat source placed at the origin, f is given by expressions (56) and (29), and there results

$$\left.\begin{aligned} f &\approx B_1' \frac{1}{\sqrt{gar}} e^{-gar(1-x/r)}(1 - x/r) \\ \text{with} \quad B_1' &= \frac{1}{4\sqrt{2\pi}} (1 + kM_\infty^2)\mu_\infty \frac{\vartheta_w}{T_o} \frac{Q}{2\lambda\vartheta_w} ga \end{aligned}\right\} \tag{62}$$

It is to be noted that an integration of f with respect to y_1 means an integration with respect to

$$v = \sqrt{ga(r_1 - x)}$$

while keeping x constant. After some simplifications,

$$\int_0^{x_1} dy_1 f(x_1,y_1) = -\frac{B_1'}{\sqrt{2}} (ga)^{-2} \frac{1}{x_1} \int_0^{\sqrt{ga(\sqrt{2}-1)x_1}} dv v^2 e^{-v^2}$$

By substituting this into formula (61), it follows that

$$\Phi_x = -\frac{\sqrt{2}}{\sigma} B_1'(ga)^{-2} \int_0^{x/2} \frac{dx_i}{x_i} \int_0^{\sqrt{ga(\sqrt{2}-1)x_1}} dv v^2 e^{-v^2}$$

and, introducing a new variable u such that

$$u = \sqrt{ga(\sqrt{2} - 1)x_1}$$

there results

$$\Phi_x = -\frac{2\sqrt{2}}{\sigma} B_1'(ga)^{-2} \int_0^{\sqrt{x_2}} \frac{du}{u} \int_0^u dv v^2 e^{-v^2} \tag{63}$$

where

$$x_2 = ga \frac{\sqrt{2} - 1}{2} x$$

By interchanging the orders of integration, there can be written

$$\Phi_x = -\frac{2\sqrt{2}}{\sigma} B_1'(ga)^{-2} \int_0^{\sqrt{x_2}} dv v^2 e^{-v^2} \int_0^{\sqrt{x_2}} \frac{du}{u}$$

$$= -\frac{2\sqrt{2}}{\sigma} B_1'(ga)^{-2} \left(\log_e \sqrt{x_2} \int_0^{\sqrt{x_2}} dv v^2 e^{-v^2} - \int_0^{\sqrt{x_2}} dv v^2 e^{-v^2} \log_e v \right) \tag{64}$$

For very small values of x_2, Φ_x is small, but it increases rapidly until a value of x_2 in the order of $1/(ga)$ is reached. Then the integrals take practically their asymptotic values which are

$$\int_0^{\sqrt{x_2} \to \infty} dv v^2 e^{-v^2} \approx \frac{\sqrt{\pi}}{4}$$

and

$$\int_0^{\sqrt{x_2} \to \infty} dv v^2 e^{-v^2} \log_e v = \text{Constant}$$

Hence formula (64) becomes

$$\Phi_x = -\frac{\sqrt{\pi}}{\sigma}\frac{B_1'}{\sqrt{2}}(ga)^{-2}\log_e\sqrt{x_2} - \text{Constant}$$

As the constant does not play a material part in the velocity perturbation, it can be disregarded, and there is obtained

$$\Phi_x = -\frac{\sqrt{\pi}}{\sigma}\frac{B_1'}{\sqrt{2}}(ga)^{-2}\log_e\sqrt{x_2}$$

and, by substituting for B_1' from expression (62),

$$\Phi_x = -\frac{1}{8\sigma}(1 + kM_\infty^2)\mu_\infty(ga)^{-1}\frac{\vartheta_w}{T_o}\frac{Q}{2\lambda\vartheta_w}\log_e\sqrt{x_2} \qquad (65)$$

Now consider a continuous distribution of heat sources over the length $4a$ of the plate, such that the intensity of the source on the segment between ξ and $\xi + d\xi$ is $q(\xi)\,d\xi$, where q has the distribution law given by formula (37a). The elementary potential perturbation $d\Phi_x$ due to $q(\xi)\,d\xi$ at a point situated at a distance x from the leading edge on the x-axis can be easily obtained from formula (65) by changing in formula (65) Q into $q(\xi)\,d\xi$ and x into $x - \xi$. Then,

$$d\Phi_x = -\frac{1}{16\sigma}(1 + kM_\infty^2)\mu_\infty(ga)^{-1}\frac{\vartheta_w}{T_o}\frac{q(\xi)\,d\xi}{2\lambda\vartheta_w}\log_e\left[ga\frac{\sqrt{2}-1}{2}(x - \xi)\right]$$

If the perturbation upstream of the source is again negligible as in the subsonic case, the perturbation Φ_x corresponding to a continuous distribution of the heat sources along the total length of the plate can be obtained by integrating the above expression between the limits 0 and x of the variable ξ. Thus,

$$\Phi_x = -\frac{1}{16\sigma}(1 + kM_\infty^2)\mu_\infty(ga)^{-1}\frac{\vartheta_w}{T_o}\int_0^x d\xi\,\frac{q(\xi)}{2\lambda\vartheta_w}\log_e\left[ga\,\frac{\sqrt{2}-1}{2}(x-\xi)\right] \tag{66}$$

For the sake of simplification in the following calculations, the integral in formula (66) will be represented by

$$S_2 = \int_0^x d\xi\,\frac{q(\xi)}{2\lambda\vartheta_w}\log_e\left[ga\,\frac{\sqrt{2}-1}{2}(x-\xi)\right] \tag{66a}$$

Generally in a supersonic flow ga is very large, so that $q(\xi)$ decreases very rapidly with ξ; therefore expression (66a) can be written in the following approximate form:

$$S_2 = \log_e\left(ga\,\frac{\sqrt{2}-1}{2}x\right)\int_0^x d\xi\,\frac{q(\xi)}{2\lambda\vartheta_w}$$

The integral of the right-hand member has been calculated in the case with formula (42) and has the value

$$\int_0^x d\xi\,\frac{q(\xi)}{2\lambda\vartheta_w} = \frac{2}{\sqrt{\pi}}\frac{1}{\mu_\infty}\sqrt{2gx}$$

where x is now the distance in absolute value from the leading edge. Hence formula (66) becomes

$$\Phi_x = -\frac{1}{8\sqrt{\pi}}\frac{1}{\sigma}(1 + kM_\infty^2)\,\frac{\vartheta_w}{T_o}(ga)^{-1}\sqrt{2gx}\log_e\left(g\,\frac{\sqrt{2}-1}{2}x\right) \tag{67}$$

The perturbation Φ_x increases in absolute value from the leading edge to the trailing edge with an average gradient

$$\overline{\frac{d\Phi_x}{dx}} = -\frac{1}{4\sqrt{\pi}} \frac{1}{\sigma} (1 + kM_\infty^2) \frac{\vartheta_w}{T_o} \frac{1}{\sqrt{8ga}} \log_e \sqrt{\frac{\sqrt{2} - 1}{2} 4ga\, U} \tag{68}$$

It is seen that in a supersonic flow the heat sources will produce a decrease in the velocity of the flow which passes over them.

NONLINEAR COOLING OF A HEATED BODY AT SUPERSONIC SPEED

Nonlinear Cooling of a Flat Plate

The role of the heat sources along a body in a compressible flow can be imagined to heat a thin layer of air which perturbs the motion of the main flow, just as if the body were deformed instead of being heated. The average velocity which passes along such a deformed body is then an "effective velocity" defined by

$$U_{eff} = U + \frac{d\Phi_x}{dx}$$

or

$$U_{eff}/U = 1 + \frac{1}{U} \frac{d\Phi_x}{dx}$$

where $d\Phi_x/dx$ is given by formula (55) in a subsonic flow, and by formula (68) in a supersonic flow. For the sake of convenience of writing, introduce

$$\epsilon = \frac{U_{eff} - U}{U} = \frac{1}{U} \frac{d\Phi_x}{dx} \tag{69}$$

as the correction coefficient of the effective velocity

$$U_{eff}/U = 1 + \epsilon$$

Introducing this effective velocity in the expression of heat delivery (43),

$$N = \frac{1}{2\mu_\infty}\left[\sqrt{8\pi ga}(1 + \epsilon/2) + 1\right] \tag{70}$$

where

$$g = \mu_\infty^{-\frac{2-k}{k-1}} hU$$

In problems of heat transfer, the following parameters play an important part:

(1) The Nusselt number N as introduced previously in formula (40) is defined by

$$Q = N\lambda A\, \vartheta_w/d$$

where Q is the amount of heat delivered in unit time by the unit length of an immersed body across an area A, ϑ_w is the difference between the temperature of the body and the isentropic stagnation temperature, and d is a representative length. For a flat plate of length $4a$, N is

$$N = \frac{Q}{2\lambda\vartheta_w}$$

(2) The Reynolds number $R = \rho_o Ud/\eta$, where U is the free-stream velocity, ρ_o is the isentropic stagnation density, and η is the coefficient of viscosity

(3) The Prandtl number $Pr = 2h\eta/\rho_o$, where $2h = \rho_o c_p/\lambda$

(4) The Peclet number $P = R \times Pr = 8hUa$

It is to be noted that if the Prandtl number is a constant, the Peclet number is proportional to the Reynolds number. If the results are to be expressed in terms of P, the quantity $8ga$ which occurs frequently in this paper becomes

$$8ga = P\mu_\infty^{-\frac{2-k}{k-1}}$$

Hence an alternative form of formula (70) for the heat delivery is obtained as follows:

$$2N = \sqrt{\pi P}\left(1 + \frac{1}{2}\epsilon\right)\mu_\infty^{-\frac{k}{2(k-1)}} + \mu_\infty^{-1} \tag{70a}$$

where ϵ is given by formulas (55) and (68) for the subsonic and the supersonic speeds, respectively, as follows:

(a) In a subsonic speed:

$$\epsilon = \frac{3\sqrt{3}}{16}\sqrt{\pi}\,\frac{1 + kM_\infty^2}{1 - M_\infty^2}\,\mu_\infty^{\frac{2-k}{2(k-1)}}\,P^{-\frac{1}{2}}\,\vartheta_w/T_o \tag{71}$$

(b) In a supersonic speed:

$$\epsilon = -\frac{1}{4}(1 + kM_\infty^2)(M_\infty^2 - 1)^{\frac{1}{2}}\,\mu_\infty^{\frac{2-k}{2(k-1)}}\,\frac{1}{\sqrt{\pi P}}\log_e\left[\sqrt{\frac{\sqrt{2}-1}{4}}\,\mu_\infty^{-\frac{2-k}{2(k-1)}}\sqrt{P}\right] \tag{72}$$

By substituting formulas (71) and (72), formula (70a) can be written also in the following form:

$$2N = C_0\sqrt{\pi P} + C_1 - \left(C_2 \log_e \sqrt{P} - C_3\right)\frac{\vartheta_w}{T_o} \tag{70b}$$

where N is the Nusselt number, P is the Peclet number (it is equal to the Reynolds number, when the Prandtl number is unity), ϑ_w is the difference between the temperature of the body and the isentropic stagnation temperature T_o, and finally C_0, C_1, C_2, and C_3 are the following functions of the free-stream Mach number M_∞ (see figs. 3 and 4).

(a) In a subsonic flow:

$$\left.\begin{aligned} C_0 &= \mu_\infty^{-\frac{k}{2(k-1)}} \\ C_1 &= \mu_\infty^{-1} \\ C_2 &= 0 \\ C_3 &= \frac{3\sqrt{3}}{32}\pi\,\frac{1 + kM_\infty^2}{1 - M_\infty^2}\,\mu_\infty^{-1} \end{aligned}\right\} \tag{73}$$

(b) In a supersonic flow:

$$\left.\begin{aligned} C_0 &= \mu_\infty^{-\frac{k}{2(k-1)}} \\ C_1 &= \mu_\infty^{-1} \\ C_2 &= \frac{1}{8}\,(1 + kM_\infty^2)(M_\infty^2 - 1)^{1/2}\mu_\infty^{-1} \\ C_3 &= C_2\left[\frac{2 - k}{2(k - 1)}\log_e \mu_\infty - \log_e\sqrt{\frac{\sqrt{2} - 1}{4}}\right] \end{aligned}\right\} \tag{73a}$$

where $\mu_\infty = 1 + \frac{k - 1}{2} M_\infty^2$, and k is the ratio of the coefficients of specific heat.

From the formula (70b) for the heat delivery of a plate, the following properties can be seen (figs. 5 and 6).

(a) Subsonic flow:

(1) N increases with ϑ_w. The rate of change does not depend on P. Since $Q = 2N\lambda\vartheta_w$, Q does not vary linearly with ϑ_w (fig. 5)

(2) N decreases as M_∞ increases

(3) N varies linearly with $\sqrt{P}$

(4) For very moderate M_∞, even for $M_\infty \approx 0$, formula (70b) reduces to

$$2N = \sqrt{\pi P} + 1 + \frac{3\sqrt{3}}{64}\,\vartheta_w/T_o$$

It reduces to the formula of King (reference 5) only if ϑ_w/T_o vanishes.

(b) Supersonic flow:

(1) N decreases as ϑ_w increases. The rate of change depends on $\log_e \sqrt{P}$ and therefore is larger in absolute value than the rate of change in a subsonic flow; Q does not vary linearly with ϑ_w (fig. 5)

(2) N decreases as M_∞ increases. This is true for a flat plate with a sharp nose. It will be seen in the following section that for a body with a blunt nose different aspects will appear

(3) N does not vary linearly with $\sqrt{P}$, because the temperature term in formula (70b) contains a function $\log_e \sqrt{P}$. However, the deviation from the linearity is small owing to the small magnitude of ϑ_w/T_o.

Some Remarks on Heat Delivery by a Body With a Blunt Nose in Supersonic Flow

In the preceding sections, a certain continuous distribution of heat sources immersed in a flow of velocity U has been considered. By the use of a generalized potential theory developed for a nonadiabatic and rotational fluid, the action of the heat sources on the flow has been studied, and the heat delivery has been calculated.

In order to show in the theory the fundamental action of heat on the flow, and to obtain the characteristic physical features, it is believed that the extraneous action of the shape of the medium, that is, the metallic body introducing heat to the fluid, must be eliminated in a first study. Therefore, in the present calculations, a fine plate has been chosen which does not offer any shape perturbation to the flow. This circumstance approaches the theoretical distribution of heat sources without interference from an extraneous body. Fundamental investigations

of the physical phenomena under such a simplified circumstance may serve as a basis for the study of the phenomena under other more complicated circumstances, for example, the action of heat combined with the action of an extraneous body. Hot-wire anemometry in an incompressible fluid takes usually a cylindrical wire for the introduction of heat to the fluid. No difficulty has ever arisen since its body perturbation is known by elementary theory. In a supersonic flow, the body perturbation is more difficult to calculate. However, for the study of the heat delivery an approximate estimation can be made by starting from the following considerations.

A blunt body like a cylinder will produce a detached shock wave which is strong on the front region and weak on the two branches. Behind the shock wave, the flow about the cylinder is subsonic on the front region and supersonic on the rest of the region. From the present calculations (see section Flux Distribution and Heat Delivery) it is known that the air in the front region gives the largest contribution in the heat delivery. That part of the air, heated under a subsonic state, passes the sonic zone and leaves the cylinder with a supersonic velocity. While the detailed picture of the phenomena has to be studied more rigorously, a rough estimation of the heat delivery can already be obtained by giving to that contributing part of the air an effective velocity after a Mach wave, and an effective subsonic Mach number M_1 which can be derived from the unperturbed Mach number M_∞ before that part of the shock wave which is approximately normal. The known relation between such Mach numbers is (see fig. 4)

$$M_1^2 = \frac{1 + \frac{k-1}{2} M_\infty^2}{kM_\infty^2 - (k-1)/2} \tag{74}$$

The temperature distribution about a cylinder is given by

$$\frac{\partial^2 \vartheta}{\partial \varphi^2} + \frac{\partial^2 \vartheta}{\partial \psi^2} = \frac{2g}{U} \frac{\partial \vartheta}{\partial \varphi} \tag{75}$$

This is the general form of equation (23). By choosing a system of coordinates φ and ψ, this distribution reduces to the temperature distribution in a uniform stream flowing along a flat plate, as given previously by equation (23). The same method used in the sections Temperature Distribution and Flux Distribution and Heat Delivery can be applied here in order to calculate the temperature distribution and

finally the heat delivery for a cylinder. Omitting the details, the result is

$$Q = \lambda \vartheta_w \mu_1^{-1} \left(\sqrt{2\pi \beta_o \, q/U} + 1 \right) \tag{76}$$

The limits of variation of the potential function are 0 and β_o. In the case of an elliptic cylinder of semiaxes (a,b),

$$\beta_o = 2(a + b)U$$

As $\beta_o = 4Ua$ for both cases of a circular cylinder of radius a and of a plate of length $4a$, it is seen that the expressions of the heat delivery for both cases without heat perturbation must be identical. It is also to be noted that, for $M_1 = 0$, the expression of the heat delivery reduces to that found by King. Further, for a circular cylinder,

$$N_c = \frac{Q}{\pi \lambda \vartheta_w}$$

$$P_c = 4hUa$$

$$(P_c)_{eff} = 4hU_{eff}a$$

Hence formula (76) becomes

$$N_c = \frac{\sqrt{2}}{\pi} C_0 \sqrt{\pi (P_c)_{eff}} + \left(\pi \mu_1 \right)^{-1} \tag{77}$$

The mass of air on the front part of the cylinder, heated at a subsonic state characterized by the subsonic Mach number M_1 will leave the cylinder with a supersonic velocity which is approximately the effective velocity after a Mach wave

$$U_{eff} = U(1 + \epsilon)$$

where ϵ is given by expression (72). By substituting expression (72) into formula (77),

$$N_c = \frac{\sqrt{2}}{\pi} C_0 \sqrt{\pi P_c} + \frac{1}{\pi} C_1 - \frac{\sqrt{2}}{\pi}\left(C_2 \log_e \sqrt{P_c} - C_3\right)\frac{\vartheta_w}{T_o} \qquad (78)$$

where the C's are the following functions of the Mach numbers:

$$C_0(M_1) = \mu_1^{-\frac{k}{2(k-1)}}$$

$$C_1(M_1) = \mu_1^{-1}$$

$$C_2(M_1) = C_2(M_\infty)\frac{C_0(M_1)}{C_0(M_\infty)}$$

$$C_3(M_1) = C_2(M_1)\left[\frac{2-k}{2(k-1)} \log_e \mu_\infty - \log_e \sqrt{\frac{\sqrt{2}-1}{4}}\right]$$

Formula (78) remains valid for a subsonic flow, provided the values of the C-functions are taken from expressions (73).

The amount of heat Q does not vary with ϑ_w (see fig. 7), although the deviation from linearity is not so important as for the case of a flat plate.

For small values of ϑ_w/T_o, the body perturbation becomes predominant compared with the heat perturbation. Therefore N_c increases with M_∞ for $M_\infty > 1$, a behavior directly opposite to that for a flat plate which does not offer any body perturbation (see fig. 8). On the other hand, for large values of ϑ_w/T_o (see fig. 9), the body perturbation becomes less important compared with the heat perturbation. Therefore N_c decreases with M_∞ for $M_\infty > 1$, a behavior comparable with that for a flat plate (cf. fig. 6).

From the above formula for the heat delivery, it is seen that Q is proportional to the coefficient $\lambda\sqrt{P}$ or $\lambda\sqrt{Pr \times Re}$. In the present theoretical investigation, this coefficient has been assumed as independent of the temperature gradient in the neighborhood of the wire, and the viscosity has been assumed negligible. In view of this simplification

and of the uncertainty attached to the value of λ, the theoretical value of the coefficient, calculated for a constant temperature of 17° C, seems too high compared with the experimental ones. By comparing the formula of King with the experiments on cylindrical hot-wires at zero Mach number, it is seen that the ratio of experimental coefficient to theoretical coefficient is in the range of 1.5 to 1.7. These considerations have to be taken into account in the numerical interpretations of the results.

Some Concluding Remarks on Heat Flow and Some Notes on Effects of Viscosity

In this section various details of the reasoning are reviewed, and the effects of viscosity are studied. This will furnish at the same time an opportunity of pointing out briefly the various principles and hypotheses underlying the theory, and the difficulties which herewith are connected.

In the present paper, it was intended to elucidate some fundamental theoretical features of a heat flow, namely,

(1) The generalized potential theory for a flow with rotation and variable heat energy

(2) The distribution of heat energy furnished by a heat source

(3) The distribution of heat flux

(4) The amount of heat delivery

(5) The perturbations of the flow by heat

A simplified model was chosen for the study; it consisted of a straight array of continuous heat sources, placed parallel to a stream of uniform velocity. The heat sources on the array were of such an intensity distribution as to furnish a prescribed surface temperature, for instance, a uniform temperature. In order to fix the idea, such an array was called an "infinitely thin flat plate," which has the only significance of indicating the location of the heat sources giving a constant surface temperature, without offering any body perturbation to the flow, for example, in the form of a boundary layer.

The theory of generalized potential flow has the advantage of permitting the use of the potential-streamline coordinates, and hence the application to more complicated shapes of heat source distribution, other than the above straight array. In fact the use of such coordinates can transform a complicated distribution into the straight array, which is therefore the most fundamental distribution.

The simplified model has served to illustrate certain physical features. However, it may, of course, deviate appreciably from an actual body supplying heat to the flow. For example, the flow in the boundary layer of the body should be actually retarded, and therefore the heat delivery should be less intense. But before going to such problems of a nonuniform stream, it was necessary first to study the more fundamental problem of a uniform stream, which gives a clearer analysis of the predominant physical characteristics. The heat delivery by a blunt body, for example, a cylinder, gives rise to additional difficulties which were not investigated, because the nonuniform flow behind a curved detached shock wave, as produced by the blunt body, is in itself also a difficult problem, even without heat introduction, and must be left out of the scope of the investigation at the present time. In that respect only some phenomena behind such a wave have been roughly stipulated in the preceding section, illustrating the difficulties which arise, rather than their explanations.

The theory of the heat flow (nonadiabatic, rotational, and with variable total energy), in connection with the above enumerated aspects, was elucidated in the preceding sections by neglecting the effects of viscosity, a procedure generally adopted in the theories of such a complicated flow. In the following lines the effects of viscosity will be estimated.

For a viscous flow, the equation of energy is (see reference 6)

$$\rho c_p \frac{DT}{Dt} - \frac{Dp}{Dt} = \lambda \nabla^2 T + \bar{\Phi}$$

where

$$\frac{D}{Dt} = \frac{\partial}{\partial t} + w \cdot \text{grad}$$

$\bar{\Phi}$ is the dissipation function (Φ is used for potential perturbation, and the symbol $\bar{\Phi}$ is therefore chosen for the dissipation function) and all other symbols have been explained in the first section of the present paper.

By substituting for p from the Navier-Stokes equation

$$\rho \frac{Dw}{Dt} = -\text{grad } p + \eta \nabla^2 w$$

where η is the coefficient of viscosity,

$$\rho \frac{D}{Dt}\left(c_p T + \frac{w^2}{2}\right) - \frac{\partial p}{\partial t} = \lambda \nabla^2 T + \left(\eta w \nabla^2 w + \bar{\Phi}\right)$$

For a stationary flow,

$$\rho w \cdot \operatorname{grad}\left(c_p T + \frac{w^2}{2}\right) = \lambda \nabla^2 T + \left(\eta w \nabla^2 w + \bar{\Phi}\right) \tag{79}$$

This partial differential equation determines the spatial distribution of the temperature T, or, following equation (1), the spatial distribution of $\vartheta(x,y)$, that is, the additional temperature furnished by the heat source. As was done in the section Small Perturbations, small perturbations will be considered by keeping only terms in ϑ of the first order, and hence it is possible to transform equation (79) into a linearized partial differential equation for ϑ. From the equation of continuity (5), and from the relation of the density-temperature ratio, it is observed that $\partial w_i/\partial x_j$ is of the same order as $\partial \vartheta/\partial x_j$ and hence the second-order terms

$$\frac{\partial w_i}{\partial x_j}\frac{\partial w_k}{\partial x_l} \quad (i,j,k,l = 1, 2, 3)$$

are negligible.

The difference between equation (79) and the energy equation (6) used formerly lies in the presence of the last expression within parentheses, which represents the effects of viscosity.

Now,

$$w \nabla^2 w = \frac{1}{2}\nabla^2 w^2 - \left(\partial w_i/\partial x_j\right)^2$$

The last term of the right-hand member is a second-order term in ϑ and is therefore negligible. So is $\overline{\Phi}$ too. Hence by neglecting second-order terms in the last expression between parentheses of equation (79),

$$\rho \mathbf{w} \cdot \operatorname{grad}\left(c_p T + \frac{w^2}{2}\right) = \lambda \nabla^2 T + \eta \nabla^2 \frac{w^2}{2}$$

$$= \frac{\lambda}{c_p} \nabla^2 \left(c_p T + \frac{\eta c_p}{\lambda} \frac{w^2}{2}\right)$$

If the Prandtl number is written as $Pr = \eta c_p / \lambda = 1$,

$$\nabla^2 \left(c_p T + \frac{w^2}{2}\right) = \frac{c_p \rho}{\lambda} \mathbf{w} \cdot \operatorname{grad}\left(c_p T + \frac{w^2}{2}\right)$$

The density ρ can be transformed into T, and the temperature T can be further transformed into ϑ, using the formulas of transformations (2) and (3). By leaving out second-order terms, after transformation there is obtained

$$\left.\begin{aligned} \nabla^2 \vartheta &= 2g' \frac{\partial \vartheta}{\partial x} \\ g' &= \mu_\infty^{-\frac{1}{k-1}} hU = g \mu_\infty^{-1} \end{aligned}\right\} \quad (80)$$

where

and g, h, μ_∞, and U are defined as usual in the first section of the paper. Equation (80) is the partial differential equation for the spatial distribution of ϑ, by taking into account the effects of viscosity. It is seen that equation (80) is of the same form as the non-viscous distribution represented by equation (23), except the coefficient $2g$ in equation (23) is replaced by $2g'$ in equation (80).

The spatial distribution of the temperature ϑ, the flux distribution, and the amount of heat delivery follow from equation (80), exactly in the same forms as formulas (29), (37a), and (43) with the new coefficient $2g'$ as follows:

$$\vartheta = \frac{Q}{2\pi\lambda} e^{g'x} K_o(g'r)\mu_\infty$$

$$q(x) = q(0)e^{2g'x}(1 - \operatorname{erf}\sqrt{2g'x})$$

with $q(0) = 2\lambda\vartheta_w 2g'/\mu_\infty$ (according to formula (41))

$$N = \frac{1}{2\mu_\infty}\left(\sqrt{8\pi g'a} + 1\right) \tag{81}$$

It is seen that the viscous distributions of ϑ and q are "rounded down," that is, not so sharp as the nonviscous distributions, and that the heat delivery is diminished by the viscosity. The effects of the viscosity, which increases with the free-stream Mach number, are thus to modify the magnitude of the physical quantities considered above, without changing the forms of their differential equations and their formulas.

From formula (81) for the heat delivery, it is seen that N is proportional to $\sqrt{g'}$, and hence to $\sqrt{U}$. Now the horizontal velocity U is subject to perturbations by the heat transfer. Those perturbations, of the order of ϑ, were assumed small, and as a first approximation they are neglected in the elucidation of formulas (43) and (81) for the heat delivery in a nonviscous flow and a viscous flow, respectively. Later, as a second approximation, the flow perturbations by heat were calculated in the sections explaining flow perturbations by heat sources at subsonic and supersonic speed, by neglecting the viscosity. It is an easy matter to substitute the perturbed velocity in the formula of unperturbed heat delivery in order to find the perturbed heat delivery. Probably the viscosity will again have certain effects on the flow perturbations. But, as the flow perturbations play only a second-order part in the heat delivery, those effects will not be studied further here.

Curves are plotted in figures for the case of a nonviscous flow, illustrating the general properties of a heat flow. The corresponding curves for a viscous flow are not added, in order not to make the figures too cumbersome.

The effects of viscosity will reduce the slope of the curves for heat delivery in figures 6, 8, and 9, without changing the intercepts with the vertical axis. The slope of the curves in figure 6 will have

to be multiplied by a factor $\mu_\infty^{-1/2}$, that is, $C_1^{1/2}$, where C_1 is given by figures 3 and 4. The slope of the curves in figures 8 and 9 will have to be multiplied by the factor

$$\mu_1^{-1/2} = \left(1 + \frac{k - 1}{2} M_1^2\right)^{-1/2}$$

where M_1 is given by figure 4 in terms of M_∞.

National Bureau of Standards
Washington, D. C., August 9, 1950

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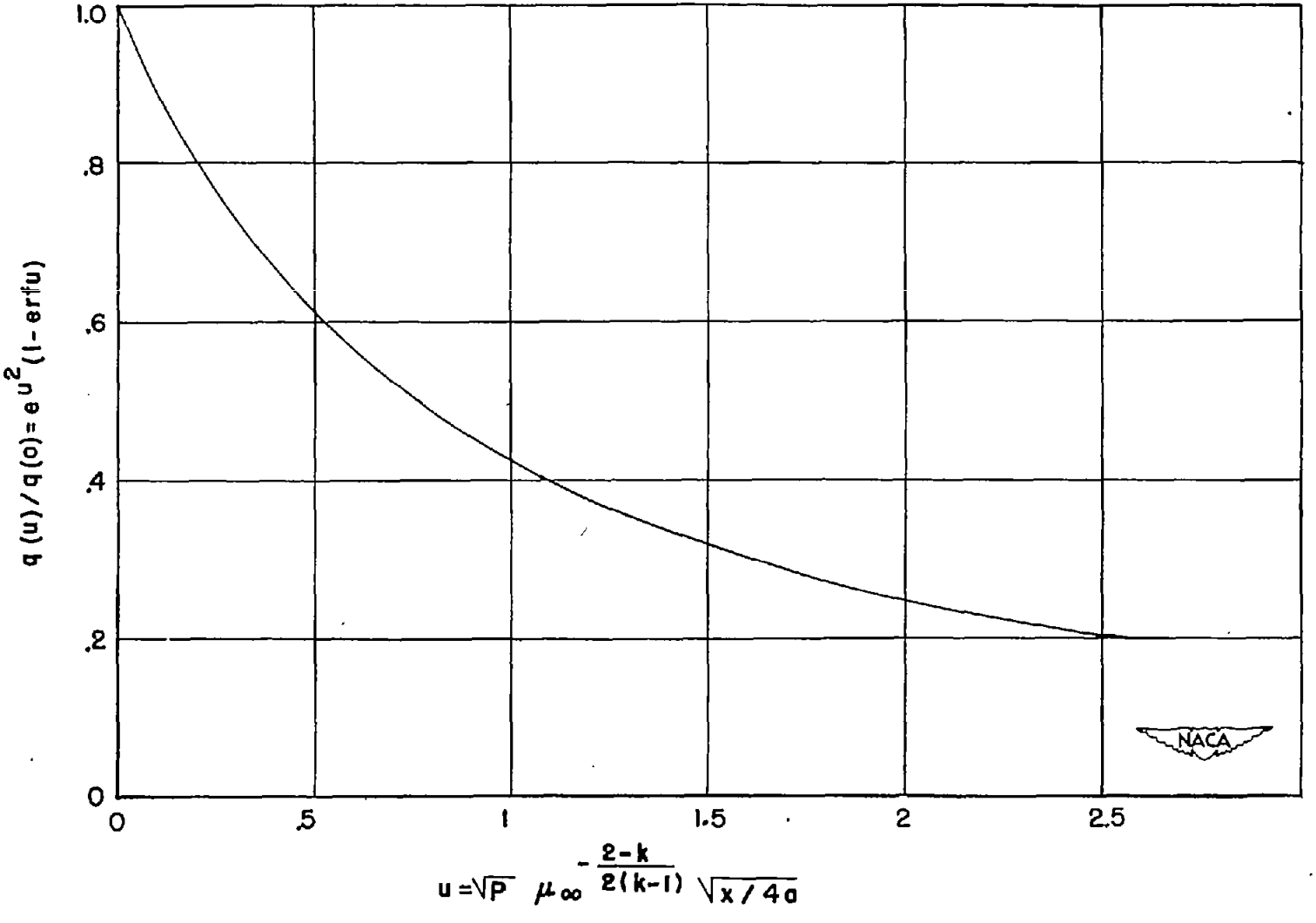

Figure 1.- Distribution of heat flux along a flat plate.

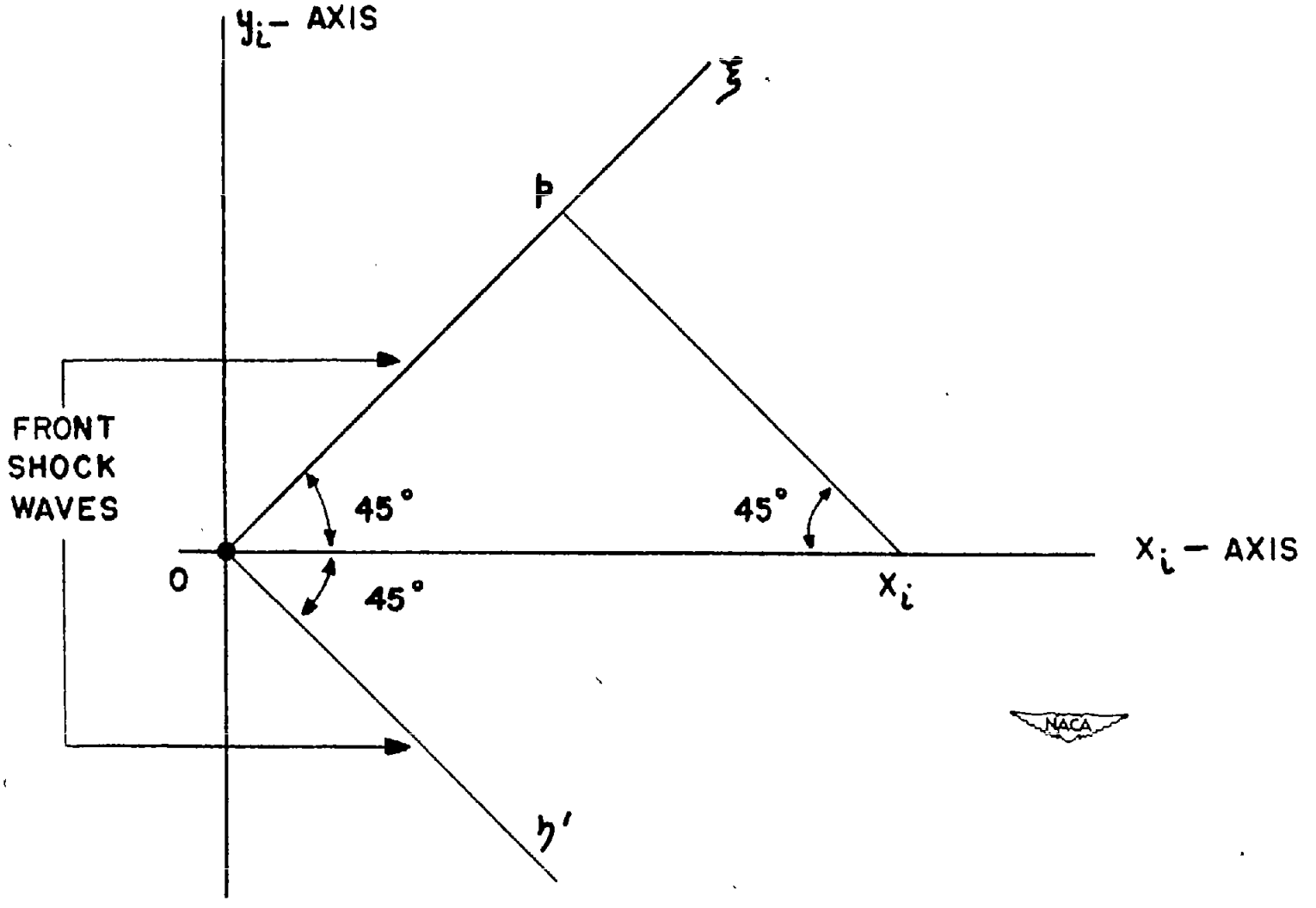

Figure 2.- Method of characteristics.

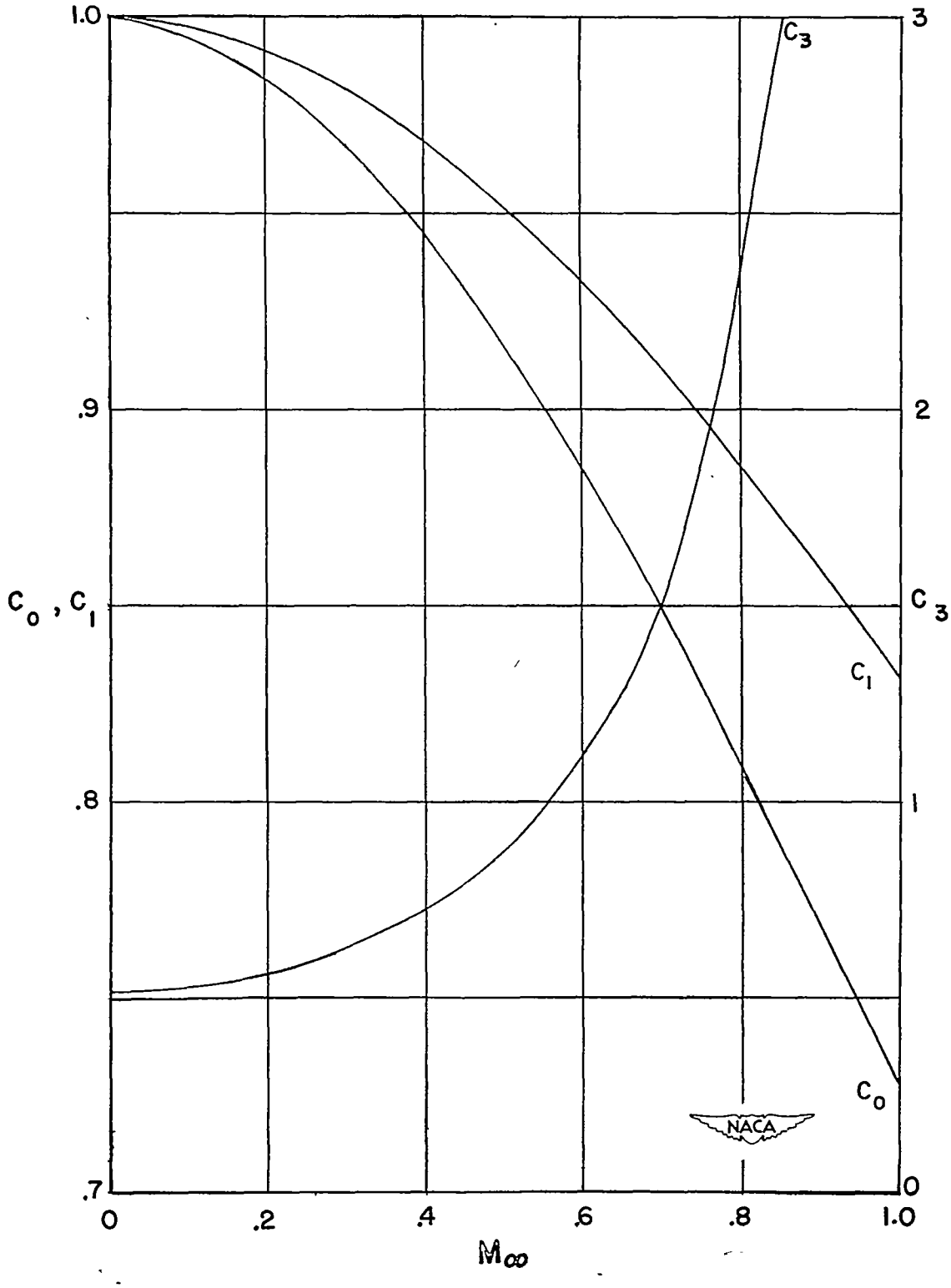

Figure 3.- Mach number functions at a subsonic speed.

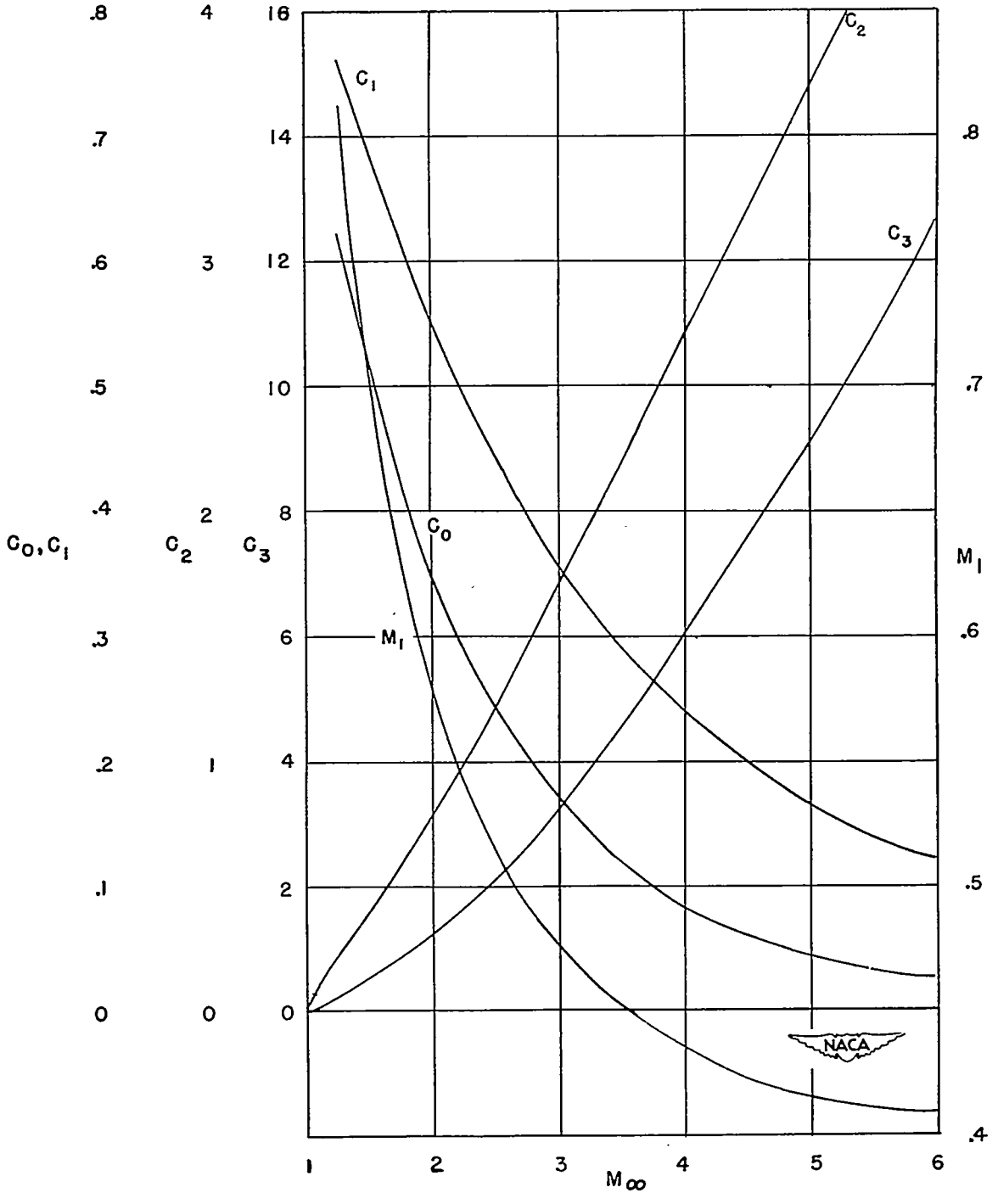

Figure 4.- Mach number functions at a supersonic speed.

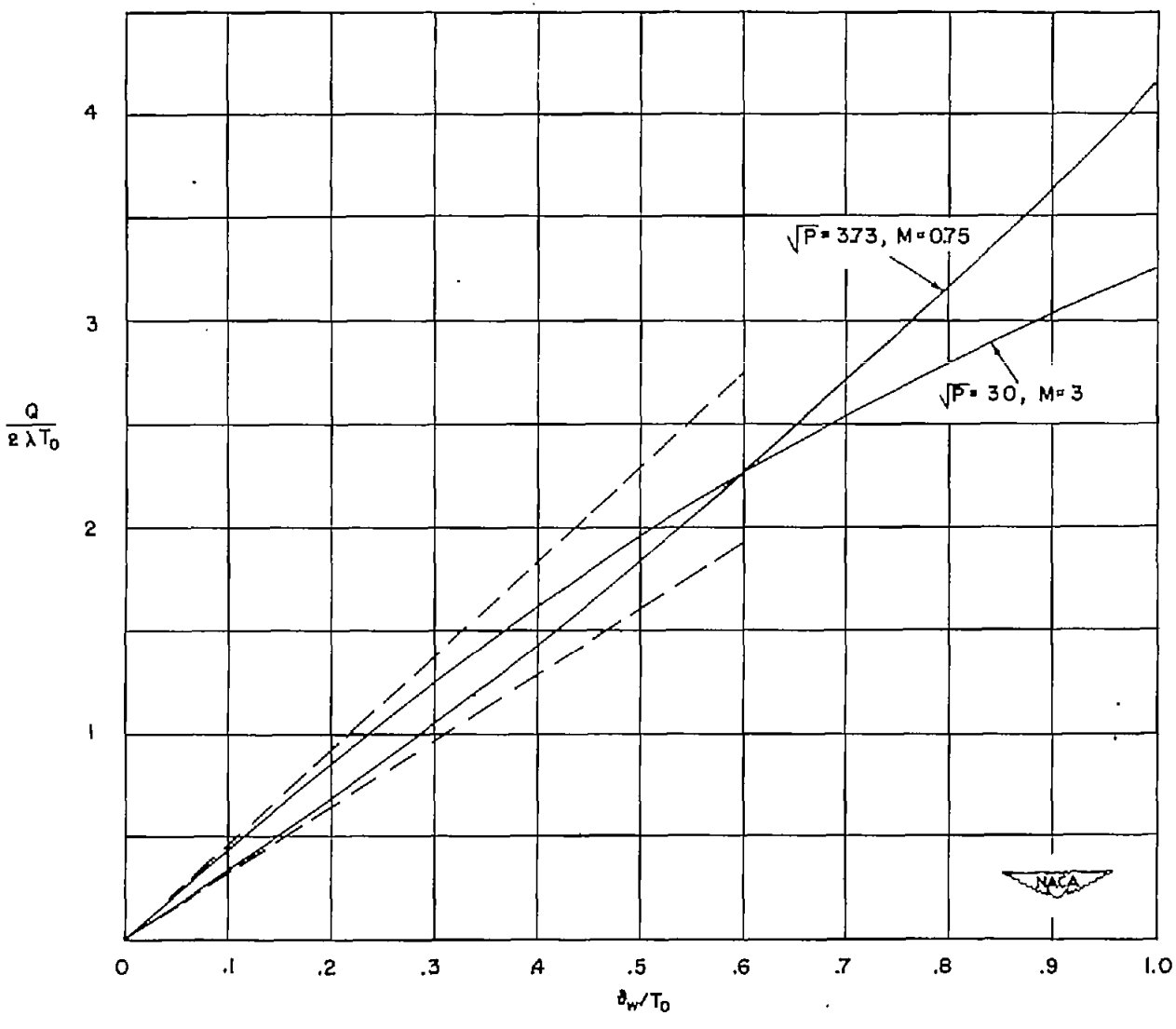

Figure 5.- Nonlinearity of heat delivery at a subsonic and a supersonic speed in the case of a flat plate. Broken lines represent corresponding linear relations obtained by neglecting heat perturbations.

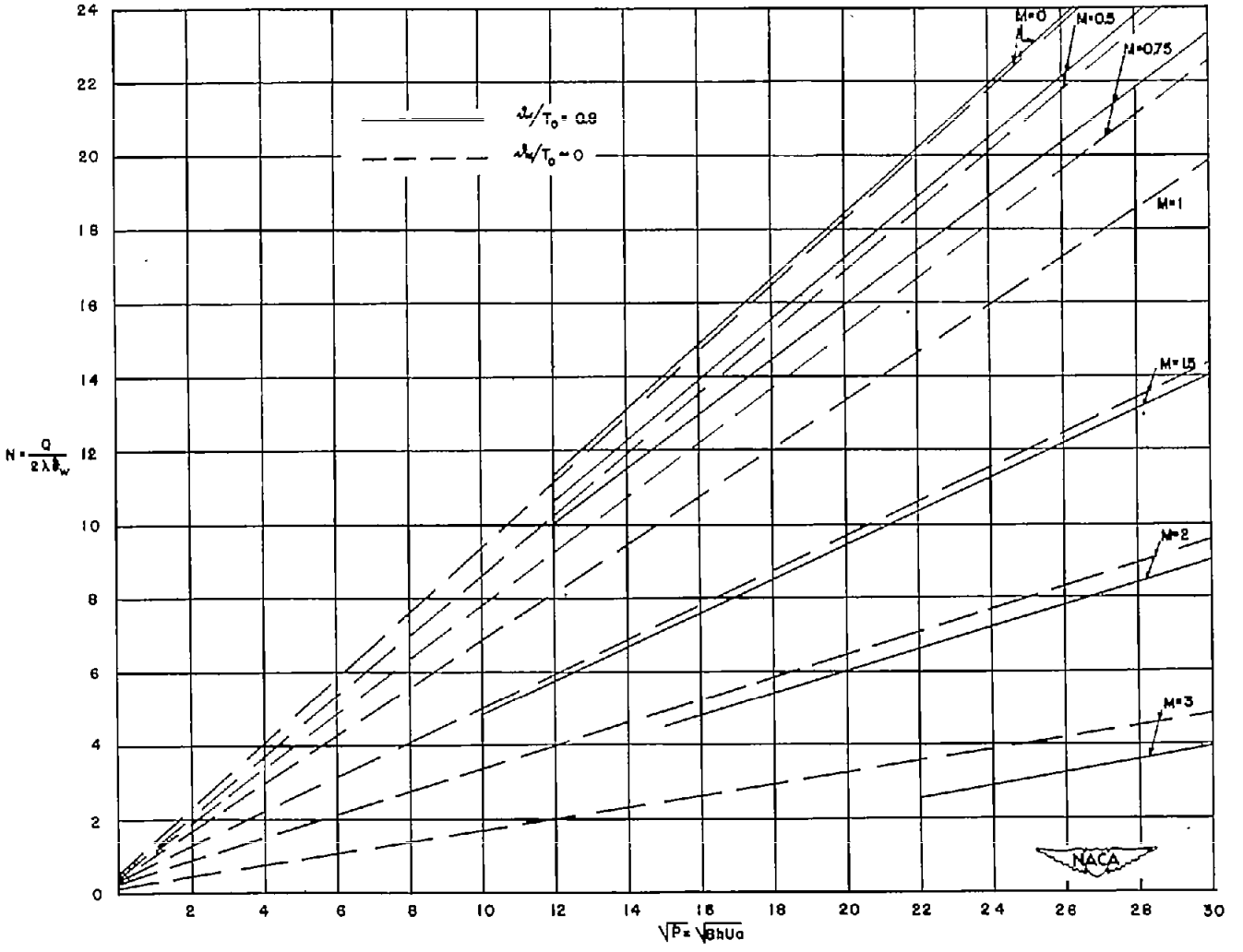

Figure 6.- Relations between heat delivery, Peclet number, Mach number, and temperature difference in the case of a flat plate. The relation for $M_\infty = 1$ and $\vartheta_w/T_0 \neq 0$ is not plotted, as it is not covered by the elliptic and hyperbolic equations of perturbations. For large Mach numbers, the theory does not predict variations of Nusselt number at low Peclet numbers.

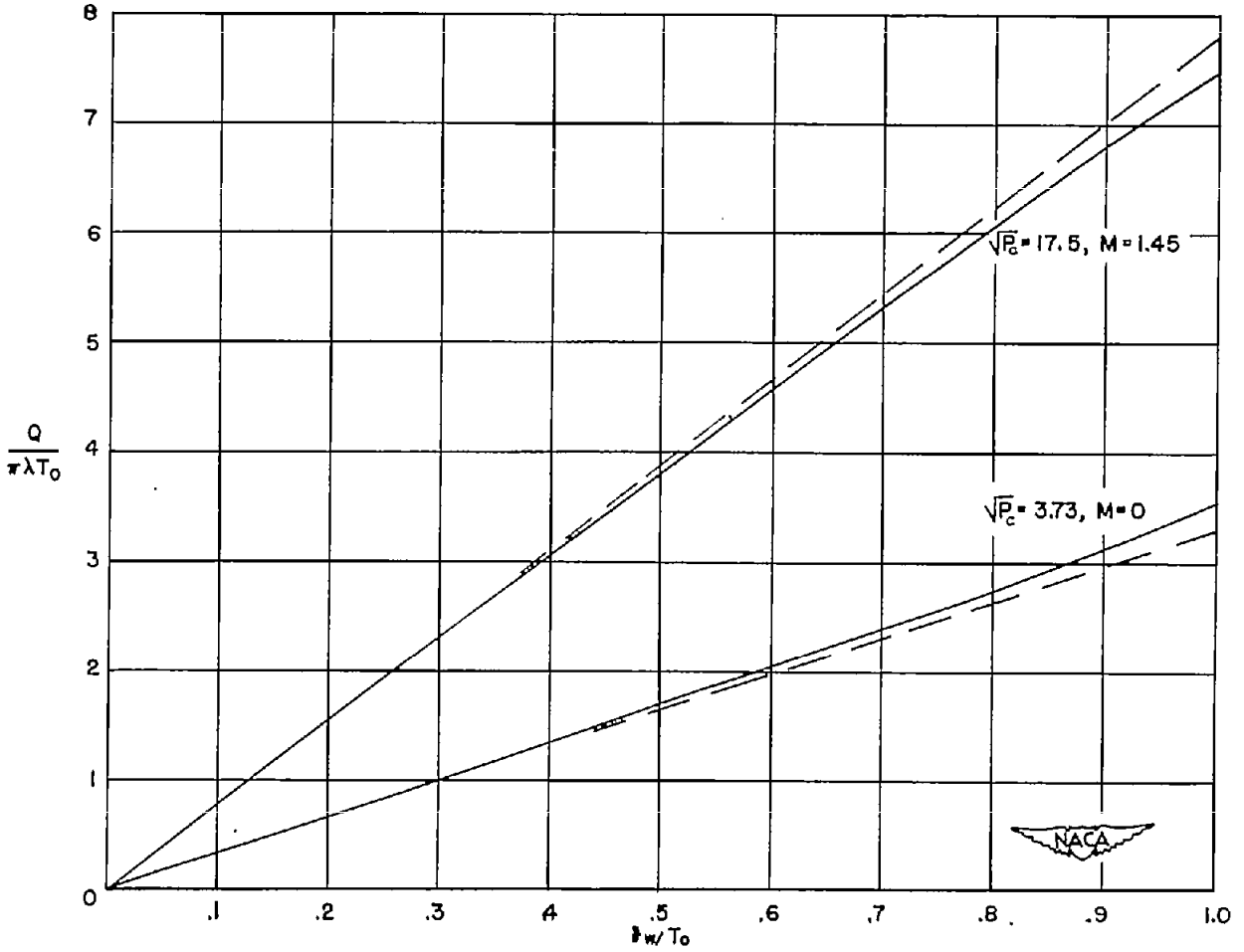

Figure 7.- Nonlinearity of heat delivery at a subsonic and a supersonic speed in the case of a cylinder. Broken lines represent corresponding linear relations obtained by neglecting heat perturbation. P_C, Peclet number for a cylinder.

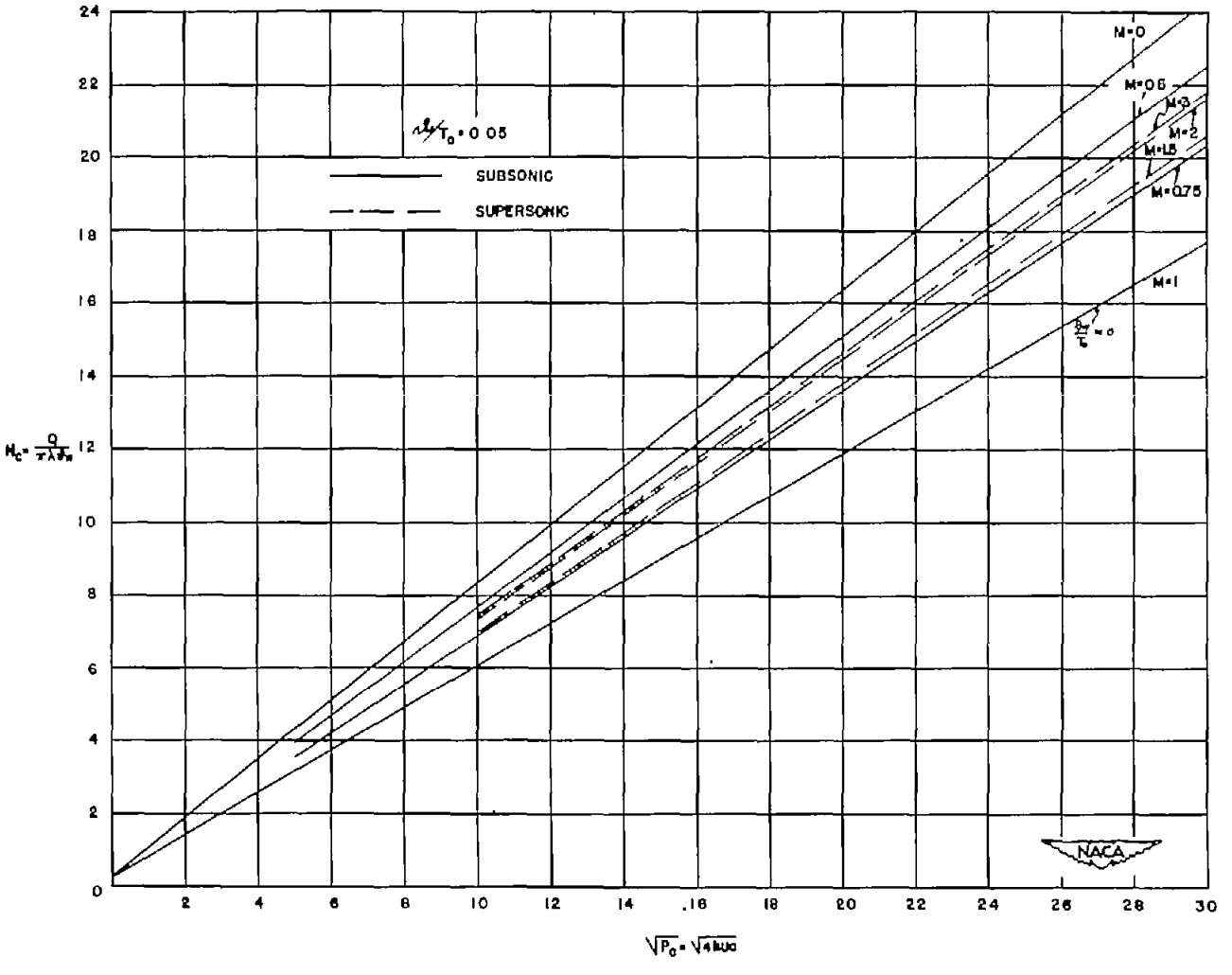

Figure 8.- Relation between heat delivery, the Peclet number, Mach number, and temperature difference in the case of a cylinder for $\vartheta_w/T_0 = 0.05$. The relation for $M_\infty = 1$ and $\vartheta_w/T_0 \neq 0$ is not plotted, as it is not covered by the elliptic and hyperbolic equations of perturbations.

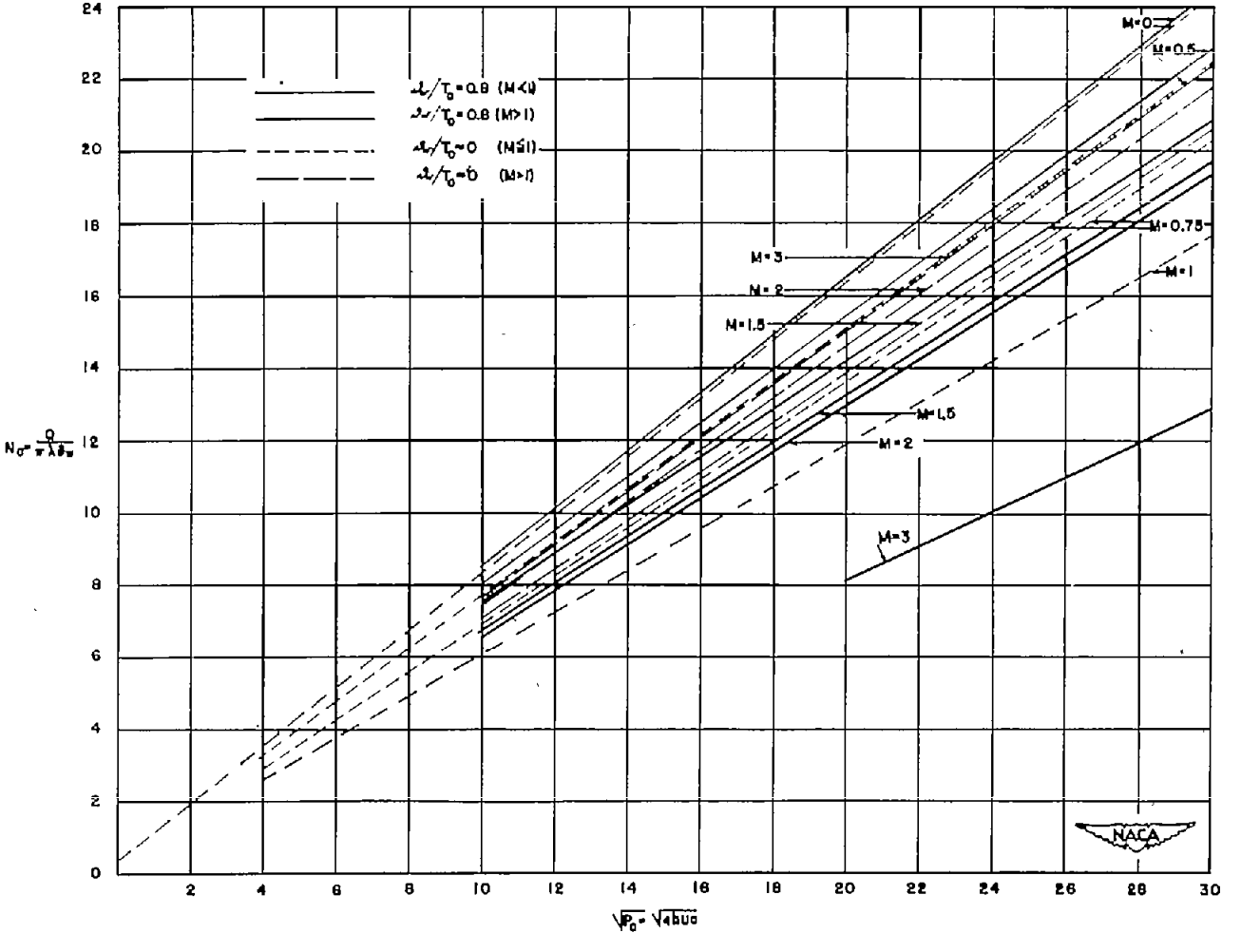

Figure 9.- Relations between heat delivery, Peclet number, Mach number, and temperature difference in the case of a cylinder for $\vartheta_w/T_0 = 0.8$. The relation for $M_\infty = 1$ and $\vartheta_w/T_0 \neq 0$ is not plotted, as it is not covered by the elliptic and hyperbolic equations of perturbations. For large Mach numbers the theory does not predict variations of Nusselt numbers at low Peclet numbers.